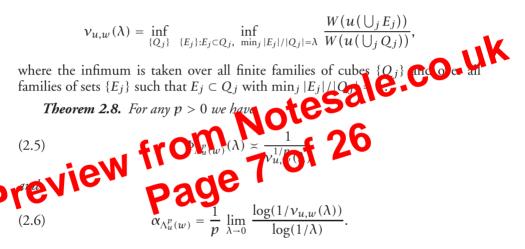
Theorem 2.7 ([5]). Let $0 . M is bounded on <math>\Lambda_u^p(w)$ if and only if there exists q < p such that for some constant c and for every finite family of cubes and sets $(Q_j, E_j)_j$ with $E_j \subset Q_j$.

(2.4)
$$\frac{W(u(\bigcup_j Q_j))}{W(u(\bigcup_j E_j))} \le c \max_j \left(\frac{|Q_j|}{|E_j|}\right)^q.$$

It is also mentioned in [5] that for a wide class of w, for instance for $w(t) = t^{\alpha}$, $\alpha > -1$, (2.4) is equivalent to the same condition but with a unique Q and $E \subset Q$. Thus, Theorem 2.7 represents a generalized version of Proposition 2.2.

We show that Theorems 2.3 and 2.4 and their proofs can be generalized with minor changes to the spaces $\Lambda_u^p(w)$. To be more precise, given weights u and w, we associate the function $v_{u,w}$ defined by



Theorem 2.9. Let 0 . Given weights u and w, the following statements are equivalent.

(i) *M* is bounded on $\Lambda_u^p(w)$;

(ii)
$$\lim_{\lambda \to 0} \frac{\nu_{u,w}(\lambda)}{\lambda^p} = +\infty;$$

(iii)
$$\lim_{\lambda \to 0} \frac{\log(1/\nu_{u,w}(\lambda))}{\log(1/\lambda)} < p;$$

(iv) if $\psi \in A$, then for any finite family of cubes $\{Q_j\}$ and any family of sets $\{E_j\}$ with $E_j \subset Q_j$,

$$\min_{j} \frac{|E_{j}|}{|Q_{j}|} \psi\left(\frac{|Q_{j}|}{|E_{j}|}\right) \leq c \left(\frac{W(u(\bigcup_{j} E_{j}))}{W(u(\bigcup_{j} Q_{j}))}\right)^{1/p}$$

The first equivalence follows from the previous proposition, and the second one is trivial.

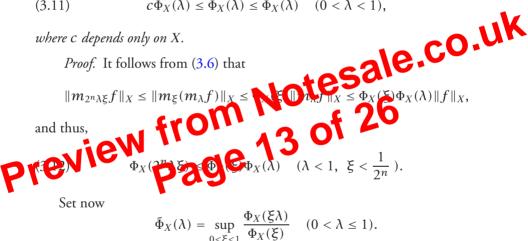
The following lemma shows that except for the trivial case $\Phi_X \equiv \infty$, Φ_X is equivalent to a finite non-increasing submultiplicative function near the origin. This is enough to give meaning to the limit in Definition 1.1 since, by Proposition 3.3, the limit defining α_X exists:

$$\alpha_X = \lim_{\lambda \to 0} \frac{\log \Phi_X(\lambda)}{\log(1/\lambda)}.$$

Lemma 3.4. Let X be any quasi-Banach function space. If $\Phi_X(\lambda_0) < \infty$, for some $\lambda_0 \in (0, 1/4^n]$ then there is a non-increasing, submultiplicative on (0, 1]function $\tilde{\Phi}_X$ such that $\tilde{\Phi}_X(1) = 1$, and

(3.11)
$$c\Phi_X(\lambda) \le \tilde{\Phi}_X(\lambda) \le \Phi_X(\lambda) \quad (0 < \lambda < 1),$$

where c depends only on X.



It is clear that $\tilde{\Phi}_X$ is submultiplicative on (0, 1] and $\tilde{\Phi}_X(1) = 1$. Next, $\tilde{\Phi}_X$ is non-increasing because Φ_X is so. Also, due to the fact that Φ_X is non-increasing, the left-hand inequality in (3.11) holds trivially with $c_1 = 1/\Phi_X(1-)$. Further, it follows from (3.12) that

$$\frac{\Phi_X(\xi\lambda)}{\Phi_X(\xi)} \le \frac{\Phi_X(\xi/2^n)}{\Phi_X(\xi)} \Phi(\lambda) \le \Phi_X(1/4^n) \Phi(\lambda),$$

which proves the right-hand inequality in (3.11) with $c_2 = \Phi_X(1/4^n)$. Observe that c_2 is finite since $\Phi_X(\lambda_0) < \infty$, $0 < \lambda_0 \le 1/4^n$.

4. PROOF OF THE MAIN RESULTS

Denote $M^2 f = MM f$. We start with the following simple lemma.

Therefore $\lim_{\lambda \to 0} \lambda \Phi_X(\lambda) = 0$, which proves (i) \Rightarrow (iv).

Assume now that (ii) holds. This means that there are constants c > 0 and $\delta < 1$ such that for any f,

(4.2)
$$\|m_{\lambda}f\|_{X} \leq c\lambda^{-\delta}\|f\|_{X}.$$

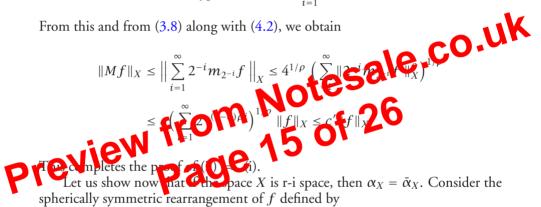
We next observe that for any cube Q,

$$\frac{1}{|Q|}\int_{Q}|f|=\int_{0}^{1}(f\chi_{Q})^{*}(\lambda|Q|)\,\mathrm{d}\lambda,$$

and hence,

$$Mf(x) \leq \int_0^1 m_\lambda f(x) \,\mathrm{d}\lambda \leq \sum_{i=1}^\infty 2^{-i} m_{2^{-i}} f(x).$$

From this and from (3.8) along with (4.2), we obtain



$$f^{\star}(x) = f^{\star}(v_n |x|^n),$$

where v_n is the volume of the unit ball. Note that the functions f and f^* are equimeasurable. It follows from (3.5) that

$$(D_{(2^n\lambda)^{1/n}}f)^{\star}(x) \le (m_{\lambda}f)^{\star}(x) \le (D_{(\lambda/3^n)^{1/n}}f)^{\star}(x).$$

Therefore,

$$\|D_{(2^n\lambda)^{1/n}}f\|_X \le \|m_\lambda f\|_X \le \|D_{(\lambda/3^n)^{1/n}}f\|_X$$

and

$$h_X\left(\frac{1}{2^n\lambda}\right) \le \Phi_X(\lambda) \le h_X\left(\frac{3^n}{\lambda}\right).$$

From this and from the definitions (1.2) and (3.9), we readily obtain that $\alpha_X = \bar{\alpha}_X.$

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