2.10. Then, $V_1V_2^{1-p}$ is again an $A_p(X)$ weight such that

$$V_1V_2^{1-p} = \left(m_E v_1 \left(m_E v_2\right)^{1-p}\right)^{\delta}$$

on E, with the maximal function m_E restricted to E as per Definition 2.2. The fact that $v_1, v_2 \in \widetilde{A}_1(E)$ implies that there is a constant $C = \max\{[\![v_1]\!]_1, [\![v_2]\!]_1\}$ such that $v_i \le m_E v_i \le C v_i$, $i = 1, 2$, almost everywhere on E (Proposition 2.3). Thus there exist nonnegative functions g_i, $i = 1, 2$, such that $g_i, g_i^{-1} \in L^\infty(X)$ and $g_i m_E v_i = v_i$ almost every where on E. Defining $g = g_1^{\delta} g_2^{\delta(p-1)}$ we see that $g, g^{-1} \in L^\infty(X)$, $g > 0$, and

$$g(x)V_1(x)V_2(x)^{1-p} = \left(v_1(x)v_2(x)^{1-p}\right)^{\delta} = v(x)^{\delta} = w(x)$$

for almost every $x \in E$. The weight $W = gV_1V_2^{1-p}$ is in $A_p(X)$ and satisfies $W = w$ a. e. on E.

Finally, if $p = 1$, we reproduce the above argument taking v_1 as v and discarding the weight v_2. □

3. Balls and chains

The aim of this section is to collect several preparatory results concerning balls in a metric space with a doubling measure. Our reason to delve into the geometry of Whitney-type balls is that they can be used to give estimates for Muckenhoupt weights over chains. In particular, Lemma 3.8 is needed to prove Lemma 4.4 in the next section, which in turn is an integral part of Holden's argument in [17]. We have found it necessary to provide an explicit proof of Lemma 3.8, as we could not locate one in the literature.

While most results in this section do not require any additional assumptions, on occasion we need to assume the existence of geodesics joining every pair of points. To cite an example of geodesic spaces relevant to partial differential equations, Corollary 8.3.16 in [16] states that a complete, doubling metric space that supports a Poincaré inequality admits a geodesic metric that is bilipschitz equivalent to the underlying metric, with constant depending on the doubling constant of the measure and the data of the Poincaré inequality.

We say that a complete metric space (X, d) is a *geodesic space* provided that any two points $x, y \in X$ can be joined by a continuous, rectifiable curve $\gamma : [a, b] \to X$ with $d(x, y) = \ell(\gamma)$, where $\ell(\gamma)$ denotes the length of γ. A rectifiable curve $\gamma : [a, b] \to X$ satisfying $\ell(\gamma) = d(\gamma(a), \gamma(b))$ is called a *geodesic* on X. Note that for a general rectifiable curve $\gamma : [a, b] \to X$, we always have the inequality $\ell(\gamma) \ge d(\gamma(a), \gamma(b))$.

We will invoke the following well-known property of geodesics: if $[a', b'] \subset [a, b]$, the subarc $\gamma_{|_{[a',b']}}$ of the geodesic $\gamma : [a, b] \to X$ is a geodesic too. Hence, for any three points $\gamma(t_i)$ on the geodesic γ such that $a \le t_0 < t_1 < t_2 \le b$, the triangle inequality for d becomes an equality:

$$d(\gamma(t_0), \gamma(t_2)) = d(\gamma(t_0), \gamma(t_1)) + d(\gamma(t_1), \gamma(t_2)).$$

Slightly abusing notation, we write $\gamma_{|_{[x_1,x_2]}}$ to mean $\gamma_{|_{[t_1,t_2]}}$ whenever $\gamma(t_i) = x_i$, $i = 1, 2$.

Throughout the rest of this section, we will assume that (X, d, μ) is a complete metric measure space such that μ satisfies the doubling condition (2). Also, when using the notation $A \approx B$ or $A \lesssim B$ for any two real numbers A, B, we understand that the constants involved may depend on the doubling constant $C_d(\mu)$.

We begin by showing two lemmas in metric geometry for future reference. In the first one, the measure does not play any role.

Lemma 3.1. *Let X be a geodesic space, and B, B' any two balls in X. Assume that* $\mathrm{rad}(B) \lesssim \mathrm{rad}(B')$ *and that B' contains the center of B. Then there exists a ball $B'' \subset B \cap B'$ with* $\mathrm{rad}(B'') \approx \mathrm{rad}(B)$.

If E_1, E_2 are subsets of D, we define $k_D(E_1, E_2) = \inf_{x_1 \in E_1, x_2 \in E_2} k_D(x_1, x_2)$. As there is no risk of ambiguity, we will leave out the subscript D in the following.

It is easy to see that the quasihyperbolic diameter of any Whitney-like ball is bounded, which is the content of the following lemma.

Lemma 3.7. *Assume further that X is a geodesic space and let $D \subset X$ be a domain. If $B \subset D$ is a ball such that $d(B, \partial D) \approx \mathrm{rad}(B)$, then $k(x, y) \le C$ for any two points $x, y \in B$.*

Proof. Let z denote the center of B, and let $\gamma \subset B$ be a rectifiable curve connecting z and x such that $\ell(\gamma|_{[z,x]}) = d(z, x)$. Then

$$k(z,x) \le \int_{\gamma|_{[z,x]}} \frac{\mathrm{d}s}{d(y, \partial D)} \lesssim \int_{\gamma|_{[z,x]}} \frac{\mathrm{d}s}{\mathrm{rad}(B)} = \frac{\ell(\gamma|_{[z,x]})}{\mathrm{rad}(B)} \le C.$$

Similarly we obtain $k(z, y) \le C$, and the triangle inequality implies $k(x, y) \le C$. $\square$

The next lemma establishes an equivalence between shortest Whitney chains and quasihyperbolic distance. It is essentially contained in the proof of Lemma 9 in [29]. For a detailed proof of the corresponding lemma in $\mathbb{R}^n$, see Proposition 6.1 in [18]. Notice that if the space X is geodesic and $D \subset X$ is a proper subset, the distance functions $d(\cdot, \partial D)$ and $d(\cdot, X \setminus D)$ coincide over D. We are then allowed to use Lemmas 3.3 and 3.4 with the distance $d(\cdot, \partial D)$ instead of $d(\cdot, X \setminus D)$.

Lemma 3.8. *Assume further that X is a geodesic space. Let $D \subset X$ be a domain and $B_i = B(x_i, r_i) \in \mathcal{W}(D)$, $i = 1, 2$. Then $\widetilde{k}(B_1, B_2) \approx k(x_1, x_2)$.*

Proof. Let $M = \widetilde{k}(B_1, B_2)$ be the length of the shortest Whitney chain joining B_1 to B_2. In the case $x_1 = x_2$, both quantities amount to zero and there is nothing to prove. Suppose now x_1 and x_2 are distinct points. First, we prove $\widetilde{k}(B_1, B_2) \lesssim k(x_1, x_2)$. Denote by γ the quasihyperbolic geodesic joining x_1 and x_2, and take z to be an arbitrary point on γ. Of all the Whitney balls containing z, we choose the one with the smallest radius, say, $B = B(x, r)$. Consider the ball B_z centered at z and with radius r. It is clear that $B_z \subset 2B$, and thus B_z is contained in D with $d(B_z, \partial D) \ge d(2B, \partial D) \ge r$ by virtue of Lemma 3.3 (ii). Also, by the properties of the Whitney decomposition (Lemma 3.3 (ii)), we have

$$d(B_z, \partial D) \le d(z, \partial D) \le d(B, \partial D) + \mathrm{diam}(B) \le 8r,$$

and we conclude that $d(B_z, \partial D) \approx \mathrm{rad}(B_z) = r$.

Let γ_z be a subarc of $\gamma \cap B_z$ passing through z and of maximal length. We claim that $\ell(\gamma_z) \ge C_1 r$ at all times. Whenever γ is not entirely contained in B_z, by the continuity of γ, there exists a point $q \in \gamma_z$ such that $d(q, z) > r/2$. Then we have $\ell(\gamma_z) \ge d(q, z) = r/2$. In the case $\gamma \subset B_z$, by the properties of the Whitney decomposition there exists a constant $0 < c < 1$ such that $\ell(\gamma_z) = \ell(\gamma) \ge d(x_1, x_2) \ge c r_1$. Furthermore, Lemma 3.4 (i) gives $r \approx r_1$ and consequently $\ell(\gamma_z) \ge C_1 r$. Recalling that $\gamma_z \subset B_z$ and $d(z, \partial D) \le 8r$, in all cases it holds that

$$\int_{\gamma_z} \frac{\mathrm{d}l}{d(y, \partial D)} \ge \frac{\ell(\gamma_z)}{r + d(z, \partial D)} \ge \frac{C_1 r}{9r} \ge C_2. \tag{12}$$

Next, we cover the geodesic γ by balls $\{B_{z_i}\}_i$, with the points $\{z_i\}_i \subset \gamma$ chosen so that every point is contained in at most two balls B_{z_i}. Among these collections we choose the one with the smallest cardinality, say $m = \#\{B_{z_i}\}$. For any $z \in \gamma$, Lemma 3.4 (ii) shows that there are at most C Whitney balls intersecting B_z. Now let M_1 be the minimal number of Whitney balls needed to cover $\bigcup_i B_{z_i}$, and denote this collection by $\mathcal{F}$. Clearly $M_1 \ge M$, because M was the length of the shortest chain joining B_1 and B_2. Also, we have that $\#\mathcal{F} = M_1$ and, by minimality, for every $B \in \mathcal{F}$ there is at

Also, by Lemma 3.8, we have $k(z_1', z_2') \approx \widetilde{k}(B_1', B_2')$ and thus $\widetilde{k}(B_1', B_2') \lesssim C_1$. With these remarks, Lemma 4.2 (ii) allows us to estimate

$$
\begin{aligned}
&\frac{1}{\mu(B_1)} \int_{B_1} w \,\mathrm{d}\mu \lesssim \left(\frac{\mu(B_1)}{\mu(B_1'')}\right)^{p-1} \frac{1}{\mu(B_1'')} \int_{B_1''} w \,\mathrm{d}\mu && (16)\\
&\lesssim \frac{\mu(B_1')}{\mu(B_1'')} \frac{1}{\mu(B_1')} \int_{B_1'} w \,\mathrm{d}\mu \\
&\lesssim \frac{1}{\mu(B_1')} \int_{B_1'} w \,\mathrm{d}\mu && (17)\\
&\lesssim \frac{1}{\mu(B_2')} \int_{B_2'} w \,\mathrm{d}\mu && (18)\\
&\lesssim \left(\frac{\mu(B_2')}{\mu(B_2'')}\right)^{p-1} \frac{1}{\mu(B_2'')} \int_{B_2''} w \,\mathrm{d}\mu && (19)\\
&\lesssim \frac{1}{\mu(B_2'')} \int_{B_2''} w \,\mathrm{d}\mu && (20)\\
&\lesssim \frac{1}{\mu(B_2)} \int_{B_2} w \,\mathrm{d}\mu.
\end{aligned}
$$

Line (16) follows from the fact that the measure $w\,\mathrm{d}\mu$ is doubling, while (18) and (19) are Lemma 4.2 (iv) and (ii), respectively. On lines (17) and (20) we used the fact that $\mu(B_i'') \approx \mu(B_i') \approx \mu(B_i)$.

Finally, if $\left(B_1' = B^0, \dots, B^N = B_2'\right)$ is the shortest Whitney chain connecting B_1' and B_2', we have that $N \lesssim C$ by the previous arguments. Since each pair of consecutive balls (B^{j-1}, B^j) in the chain has nonempty intersection, we have $\operatorname{rad}(B^{j-1}) \approx \operatorname{rad}(B^j)$ by Lemma 3.3(iv) and therefore $\operatorname{rad}(B^0) \approx \operatorname{rad}(B^j) \approx \operatorname{rad}(B^N)$ for every for every $j = 1, \dots, N$, because $N \lesssim C$. Moreover, if $p_j \in B^{j-1} \cap B^j$, the triangle inequality gives

$$
d(p_0, p_N) \le \sum_{j=1}^{N} d(p_{j-1}, p_j) \le \sum_{j=1}^{N} 2\operatorname{rad}(B_{j-1}) \lesssim \operatorname{rad}(B^N).
$$

It follows from Lemma 3.2 that $\mu(B_1') \approx \mu(B_2')$, which in turn implies $\mu(B_1) \approx \mu(B_2)$. We conclude that

$$
\int_{B_1} w \,\mathrm{d}\mu \lesssim \int_{B_2} w \,\mathrm{d}\mu
$$

and, swapping the roles of B_1 and B_2, the inequality in the other direction. $\square$

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