Proof

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Simulation Lecture Notes

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1 Simulation From a Known Distribution

Assumptions We know how to simulate from U(0,1).

$$x_1, ..., x_n \sim U(0,1), i.i.d.$$

Psudo random numbers: $r_1, ...r_k, ...$

Set $x_i = \frac{r_i mod M}{M}$, where M is a big constant number.
E.g. M=50000. If $y_i < M, x_i = \frac{y_i}{M}$
If $y_i > 50000, y_i = 62503 \Rightarrow x_i = \frac{12503}{50000}$
If $y_i = 262525 = 5 \times 50000 + 12525 \Rightarrow x_i = \frac{12525}{50000}$

A. Simulate from an exponential distribution $x \sim exp(\lambda)$

Density function: $f(x) = \lambda e^{-\lambda x} \mathbf{1}_{\{x>0\}}$

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Algorithm

1. Simulate $u \sim U(0,1)$
2. Compute $x = -\frac{1}{\lambda} log(u)$

Claim: $x \sim f(x)$

Reason $\forall t > 0$,

$$\begin{aligned}
P(x \leq t) &= P(-\frac{1}{\lambda} log(u) \leq t) \\
&= P(u \geq e^{-\lambda t}) \\
&= 1 - P(u < e^{-\lambda t}) \\
&= 1 - e^{-\lambda t)} \\
&= \int_0^t \lambda e^{-\lambda s} ds \\
&= \int_0^t f_{exp}(s) ds
\end{aligned}$$

B. Simulate from a known continuous CDF $x \sim F(x)$

Facts Suppose

$$X \sim F(x) \Rightarrow F(X) \sim U(0,1)$$
$$U \sim U(0,1) \Rightarrow X = F^{-1}(U) \sim F(x)$$

Proof $\forall 0 < t < 1$,

$$\begin{aligned} F_u(t) &= P(U \leq t) \\ &= P(F(X) \leq t) \\ &= P(X \leq F^{-1}(t)) \\ &= F(F^{-1}(t)) = t \end{aligned}$$

$$\begin{aligned} P(X \leq x) &= P(F^{-1}(u) \leq x) \\ &= P(u \leq F(x)) \\ &= F(x) \end{aligned}$$

Algorithm

:
1. Simulate $u \sim U(0,1)$
2. Compute $x = F^{-1}(u)$

Claim: $x \sim F(x)$

E.g. Set $F(x) = 1 - e^{-\lambda x} = t$. Then $x = -\frac{1}{\lambda} log(1-t) = F^{-1}(t)$

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C. Simulate from normal distribution $x \sim N(0,1)$

Density Function: $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$
CDF: $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$

If you have a way to compute $\Phi(x)$ or $\Phi^{-1}(t)$, then you can use the Algorithm in 1.2. But there are faster methods!

Algorithm 1(approx method)

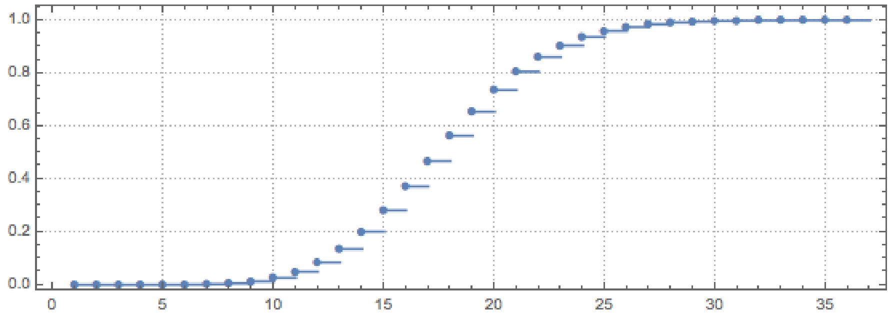
1. Simulate $u_1, u_2, ..., u_{12} \sim U(0,1)$
2. Compute $x = u_1 + ... + u_{12} - 6$

Claim: $X \sim N(0,1)$

Simulate $u \sim U(0,1)$. Compute $t = -\frac{log u}{\lambda}$
Set $k = k+1$ and $s = s+t$

3. Return $x = k-1$

Alternatively, recall $U \sim U(0,1) \Rightarrow X = F^{-1}(u) \sim F(\bullet)$



Q: In discrete case, how to define $F^{-1}(u)$?
A: $x = F^{-1}(u)$ =the largest integer such that $F(x) \leq u$

Algorithm(numerical)

1. Set $f = e^{-\lambda}, F = 0, X = 0$.
   Generate $u \sim U(0,1)$
2. While $(F \leq u)$, set
   $x = x+1$
   $f = f\frac{\lambda}{x}$
   $F = F+f$
3. Return x

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Here, F(x,y) is the joint CDF of (X,Y). $F_1(x)$ is the marginal distribution of X, $F_2(y)$ is the marginal distribution of Y.

Set $x = F_1^{-1}(t), y = F_2^{-1}(s)$,

$$F(x,y) = C(F_1(x), F_2(y))$$

Implies: if I know marginal distribution $F_1(x)$ and $F_2(y)$ and copulas $C(t,s)$, I will know the joint distribution F(x,y). Vice versa, since $F_1(x) = \int F(x,y)dy, F_2(y) = \int F(x,y)dx$.

Special cases

(a) X an Y independent $\Leftrightarrow F(X,Y) = F_1(X)F_2(Y) \Leftrightarrow C(t,s) = F_1(F_1^{-1}(t))F_2(F_2^{-1}(s)) = ts$

(b) Suppose

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right)$$

What is C(t,s)?

$F_1(x) = P(X \leq x) = P(\frac{X-\mu_1}{\sigma_1} \leq \frac{x-\mu_1}{\sigma_1}) = \Phi(\frac{x-\mu_1}{\sigma_1})$

$F_2(y) = \Phi(\frac{y-\mu_2}{\sigma_2})$

Joint density:

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}\right]}$$

Joint CDF:

$$F(x,y) = P(X, Y \leq y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v)dudv =$$

$$\begin{aligned} C(t,s) =& F(F_1^{-1}(t), F_2^{-1}(s)) \\ =& \int_{-\infty}^{\mu_2+\sigma_2\Phi^{-1}(s)} \int_{-\infty}^{\mu_1+\sigma_1\Phi^{-1}(t)} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(u-\mu_1)^2}{\sigma_1^2} + \frac{(v-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(u-\mu_1)(v-\mu_2)}{\sigma_1\sigma_2}\right]} dudv \\ =& \int_{-\infty}^{\Phi^{-1}(s)} \int_{-\infty}^{\Phi^{-1}(t)} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}[u'^2+v'^2-2\rho u'v']} du'dv' \end{aligned}$$

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When $\rho = 0$,

$$\begin{aligned} C(t,s) =& \int_{-\infty}^{\Phi^{-1}(s)} \int_{-\infty}^{\Phi^{-1}(t)} \frac{1}{2\pi} e^{-\frac{1}{2}[u'^2+v'^2]} du'dv' \\ =& \int_{-\infty}^{\Phi^{-1}(t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u'^2} du' \int_{-\infty}^{\Phi^{-1}(s)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v'^2} dv' \\ =& \Phi(\Phi^{-1}(t))\Phi(\Phi^{-1}(s)) = ts \end{aligned}$$

(c) Relating Kendall's τ to copula, we can prove that $\tau_k = 4\int\int C(t,s)dC(t,s) - 1$, where $C(t,s)$ is the copula of (X,Y).

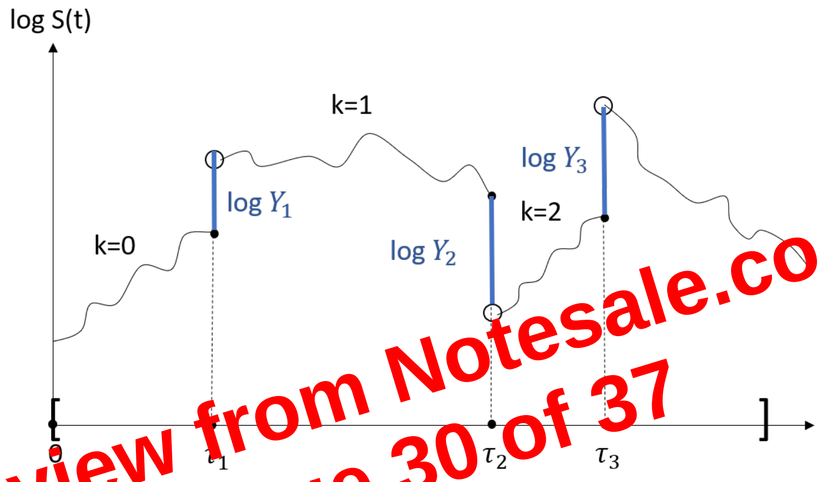
Algorithm to simulate S(t) for $0 = t_0 < t_1 < ... < t_n = T$

Set initial S(0). For $i = 1, ..., n$,

$$S(t_i) = S(t_{i-1})e^{(\mu-\sigma^2/2)(t_i-t_{i-1})+\sigma(t_i-t_{i-1})Z_i} \prod_{j=1}^{N(t)} Y_j$$

where

- $Z_i \sim N(0,1)$
- jump times $D \sim Poisson(\lambda(t_i - t_{i-1}))$
- total times at t_i $N(t_i) = D + N(t_{i-1})$
- jump variation from t_{i-1} to t_i $Y_{N(t_{i-1})+1}, ..., Y_{N(t_{i-1})+D=Y_{N(t_i)}} \sim lognormal$



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Alternatively, we can simulate the jump exactly.

$$S(\tau_{k+1}-) = S(\tau_k)e^{(\mu-\sigma^2/2)(\tau_{k+1}-\tau_k)+\sigma[W(\tau_{k+1})-W(\tau_k)]}$$

$$S(\tau_{k+1}) = S(\tau_{k+1}-)Y_k$$

where waiting time $R_k \sim exp(\lambda) : \tau_{k+1} = \tau_k + R_k$, $Y_k \sim lognormal$.
Within each (τ_k, τ_{k+1}), we can just use the regular way to simulate from GBM(μ, σ^2) for the grids.