Hence, equating LHS and RHS, we get $k = 1$.

Example 15 If z_1 and z_2 both satisfy $z + \bar{z} = 2|z-1|$ arg $(z_1 - z_2) = \frac{\pi}{4}$, then find Im $(z_1 + z_2)$.

Solution Let $z = x + iy$, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Then $\quad z + \bar{z} = 2|z-1|$

$\Rightarrow \quad (x + iy) + (x - iy) = 2\,|x-1+iy|$

$\Rightarrow \quad 2x = 1 + y^2 \qquad$ ... (1)

Since z_1 and z_2 both satisfy (1), we have

$2x_1 = 1 + y_1^{\,2}$ ... and $2x_2 = 1 + y_2^{\,2}$

$\Rightarrow \quad 2\,(x_1 - x_2) = (y_1 + y_2)\,(y_1 - y_2)$

$\Rightarrow \quad 2 = (y_1 + y_2)\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \qquad$ ... (2)

Again $\quad z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$

Therefore, $\quad \tan\theta = \frac{y_1 - y_2}{x_1 - x_2}$, where $\theta = \arg(z_1 - z_2)$

$\Rightarrow \quad \tan\frac{\pi}{4} = \frac{y_1 - y_2}{x_1 - x_2} \qquad \left(\text{since } \theta = \frac{\pi}{4}\right)$

i.e., $\quad 1 = \frac{y_1 - y_2}{x_1 - x_2}$

From (2), we get $2 = y_1 + y_2$, i.e., Im $(z_1 + z_2) = 2$

Objective Type Questions

Example 16 Fill in the blanks:

(i) The real value of 'a' for which $3i^3 - 2ai^2 + (1 - a)i + 5$ is real is ________.

(ii) If $|z| = 2$ and arg $(z) = \frac{\pi}{4}$, then z = ________.

(iii) The locus of z satisfying arg $(z) = \frac{\pi}{3}$ is _______.

(iv) The value of $(-\sqrt{-1})^{4n-3}$, where $n \in \mathbf{N}$, is _______.

(v) The conjugate of the complex number $\frac{1-i}{1+i}$ is _____.

(vi) If a complex number lies in the third quadrant, then its conjugate lies in the ______.

(vii) If $(2 + i)(2 + 2i)(2 + 3i) \ldots (2 + ni) = x + iy$, then $5.8.13 \ldots (4 + n^2) =$ ______.

Solution

(i) $3i^3 - 2ai^2 + (1 - a)i + 5 = -3i + 2a + 5 + (1 - a)i$

$= 2a + 5 + (-a - 2)\,i$, which is real if $-a - 2 = 0$ i.e. $a = -2$.

(ii) $z = |z|\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\,(1+i)$

(iii) Let $z = x + iy$. Then its polar form is $z = r(\cos\theta + i\sin\theta)$, where $\tan\theta = \frac{y}{x}$ and θ is arg (z). Given that $\theta = \frac{\pi}{3}$. Thus.

$\tan\frac{\pi}{3} = \frac{y}{x} \Rightarrow y = \sqrt{3}x$, where $x > 0$, $y > 0$.

Hence, locus of z is the part of $y = \sqrt{3}x$ in the first quadrant except origin.

(iv) Here $(-\sqrt{-1})^{4n-3} = (-i)^{4n-3} = (-i)^{4n}\,(-i)^{-3} = \frac{1}{(-i)^3}$

$$= \frac{1}{-i^3} = \frac{1}{i} = \frac{i}{i^2} = -i$$

(v) $\frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{1+i^2-2i}{1-i^2} = \frac{1-1-2i}{1+1} = -i$

Hence, conjugate of $\frac{1-i}{1+i}$ is i.

(vi) Conjugate of a complex number is the image of the complex number about the x-axis. Therefore, if a number lies in the third quadrant, then its image lies in the second quadrant.

(vii) Given that $(2 + i)(2 + 2i)(2 + 3i) \ldots (2 + ni) = x + iy$ ... (1)

$\Rightarrow$ $\overline{(2+i)}\;\overline{(2+2i)}\;\overline{(2+3i)} \ldots \overline{(2+ni)} = \left(\overline{x+iy}\right) = (x - iy)$

i.e., $(2 - i)(2 - 2i)(2 - 3i) \ldots (2 - ni) = x - iy$ ... (2)

Example 18 Match the statements of column A and B.

	Column A		Column B
(a)	The value of $1+i^2+i^4+i^6+\dots i^{20}$ is	(i)	purely imaginary complex number
(b)	The value of i^{-1097} is	(ii)	purely real complex number
(c)	Conjugate of $1+i$ lies in	(iii)	second quadrant
(d)	$\frac{1+2i}{1-i}$ lies in	(iv)	Fourth quadrant
(e)	If $a, b, c \in$ R and $b^2-4ac<0$, then the roots of the equation $ax^2+bx+c=0$ are non real (complex) and	(v)	may not occur in conjugate pairs
(f)	If $a, b, c \in$ R and $b^2-4ac>0$, and b^2-4ac is a perfect square, then the roots of the equation $ax^2+bx+c=0$	(vi)	may occur in conjugate pairs

Solution

(a) $\Leftrightarrow$ (ii), because $1+i^2+i^4+i^6+\dots+i^{20}$
$= 1-1+1-1+\dots+1 = 1$ (which is purely a real complex number)

(b) $\Leftrightarrow$ (i), because $i^{-1097} = \frac{1}{(i)^{1097}} = \frac{1}{i^{4\times 274+1}} = \frac{1}{\{(i)^4\}^{274}(i)} = \frac{1}{i} = \frac{i}{i^2} = -i$

which is purely imaginary complex number.

(c) $\Leftrightarrow$ (iv), conjugate of $1+i$ is $1-i$, which is represented by the point $(1, -1)$ in the fourth quadrant.

(d) $\Leftrightarrow$ (iii), because $\frac{1+2i}{1-i} = \frac{1+2i}{1-i}\times\frac{1+i}{1+i} = \frac{-1+3i}{2} = -\frac{1}{2}+\frac{3}{2}i$, which is represented by the point $\left(-\frac{1}{2}, \frac{3}{2}\right)$ in the second quadrant.

(e) $\Leftrightarrow$ (vi), If $b^2-4ac<0 = D<0$, i.e., square root of D is a imaginary number, therefore, roots are $x = \frac{-b \pm \text{Imaginary Number}}{2a}$, i.e., roots are in conjugate pairs.

Example 23 What is the conjugate of $\dfrac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}$?

Solution Let

$$z = \frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}} \times \frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}+\sqrt{5-12i}}$$

$$= \frac{5+12i+5-12i+2\sqrt{25+144}}{5+12i-5+12i}$$

$$= \frac{3}{2i} = \frac{3i}{-2} = 0-\frac{3}{2}i$$

Therefore, the conjugate of $z = 0 + \dfrac{3}{2}i$

Example 24 What is the principal value of amplitude of $1 - i$?

Solution Let θ be the principle value of amplitude of $1 - i$. Since

$$\tan\theta = -1 \Rightarrow \tan\theta = \tan\left(-\frac{\pi}{4}\right) \Rightarrow \theta = -\frac{\pi}{4}$$

Example 25 What is the polar form of the complex number $(i^{25})^3$?

Solution $z = (i^{25})^3 = (i)^{75} = i^{4\times18+3} = (i^4)^{18}\,(i)^3$

$= i^3 = -i = 0 - i$

Polar form of $z = r(\cos\theta + i\sin\theta)$

$$= 1\left\{\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right\}$$

$$= \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$$

Example 26 What is the locus of z, if amplitude of $z - 2 - 3i$ is $\dfrac{\pi}{4}$?

Solution Let $z = x + iy$. Then $z - 2 - 3i = (x - 2) + i(y - 3)$

Let θ be the amplitude of $z - 2 - 3i$. Then $\tan\theta = \dfrac{y-3}{x-2}$

$$\Rightarrow \qquad \tan\frac{\pi}{4} = \frac{y-3}{x-2}\left(\text{since } \theta = \frac{\pi}{4}\right)$$