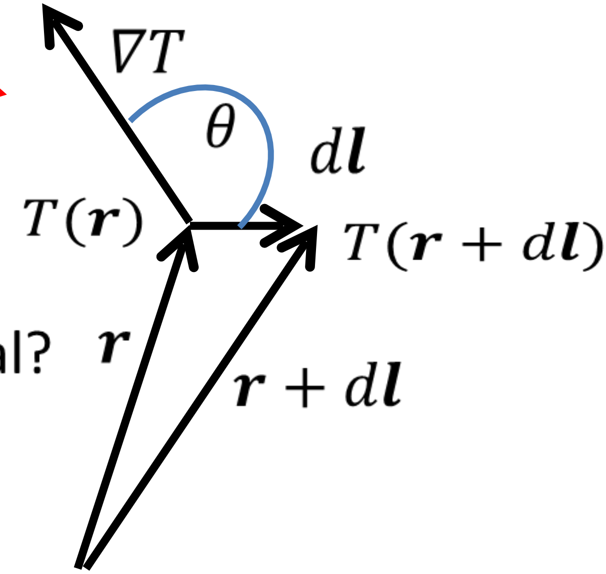
$\boldsymbol{\nabla} T \equiv \left(\frac{\partial T}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial T}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial T}{\partial z}\hat{\boldsymbol{z}}\right)$ is called the gradient of the scalar field



$dT = \boldsymbol{\nabla} T \cdot d\boldsymbol{l} = |\nabla T||d\boldsymbol{l}\,| cos\theta$

Along which direction is the variation of T maximal?

Variation of T is maximal along direction of $\boldsymbol{\nabla} T$

What does the magnitude of $\boldsymbol{\nabla} T$ represent along that direction?

$|\boldsymbol{\nabla} T|$ is the 'slope' of T along direction of $\boldsymbol{\nabla} T$: $|\nabla T| = \frac{dT}{dl}$

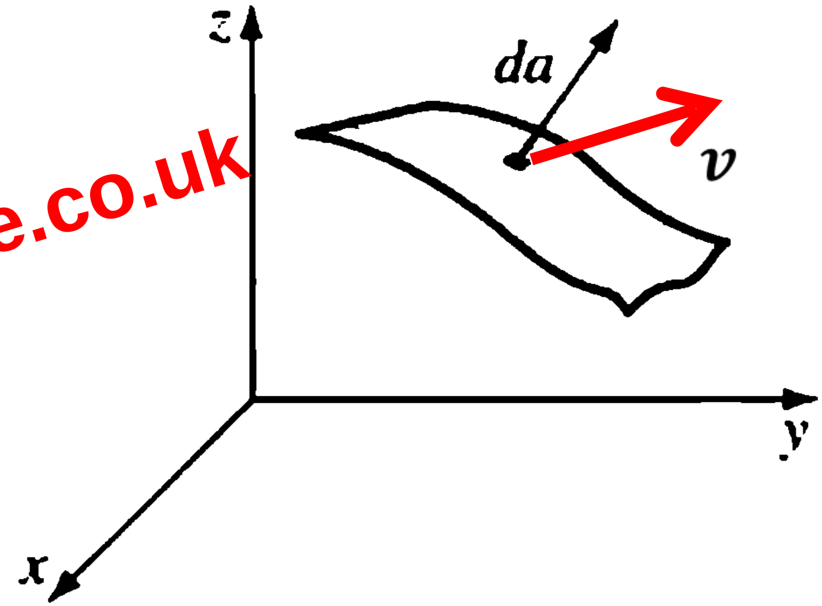
There may be points at which $\boldsymbol{\nabla} T = 0$. What happens at those points?

Surface integrals

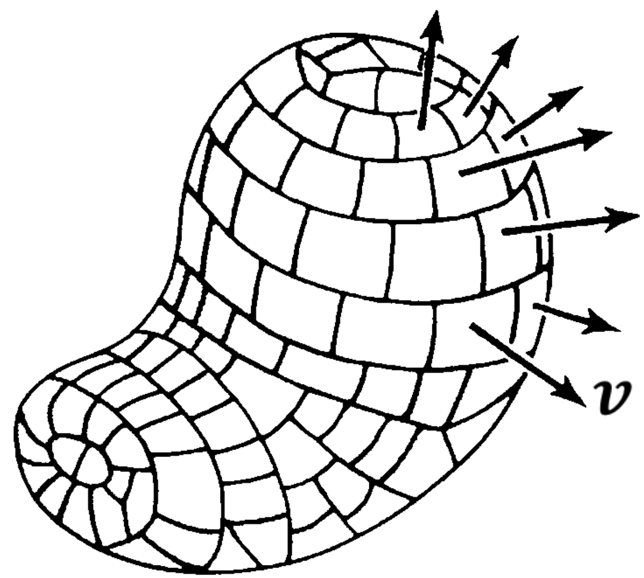
Any surface:: $\int_S \boldsymbol{v} \cdot d\boldsymbol{a}$

$\int_S \boldsymbol{v} \cdot d\boldsymbol{a}$ is called flux of $\boldsymbol{v}$ through S

Closed surface: $\oint \boldsymbol{v} \cdot d\boldsymbol{a}$

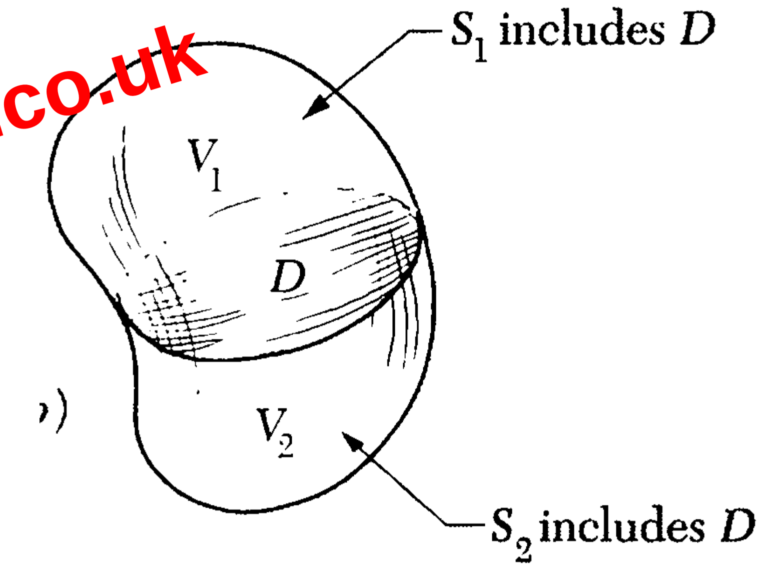


Closed surface: $d\boldsymbol{a}$ can be chosen to point towards exterior or to point towards interior. Will the flux change? How?



Additivity

$$\oint_S \boldsymbol{v} \cdot d\boldsymbol{a} = \oint_{S_1} \boldsymbol{v} \cdot d\boldsymbol{a} + \oint_{S_2} \boldsymbol{v} \cdot d\boldsymbol{a}$$



Example 1.7. Calculate the surface integral of $\mathbf{v} = 2xz\,\hat{\mathbf{x}} + (x+2)\,\hat{\mathbf{y}} + y(z^2-3)\,\hat{\mathbf{z}}$ over five sides (excluding the bottom) of the cubical box (side 2) in Fig. 1.23. Let "upward and outward" be the positive direction, as indicated by the arrows.

$$\int_S (\nabla \times \boldsymbol{v}) \cdot d\boldsymbol{a} = \oint_C \boldsymbol{v} \cdot d\boldsymbol{l}$$

What is the meaning of curl?

Take some flat surface. If you can show that the circulation along the corresponding closed curve is different from zero it means that there are points (regions) on the surface for which $\nabla \times \boldsymbol{v} \neq \boldsymbol{0}$.

Take a point P on surface. Make surface smaller and smaller around P. Analyze Stokes theorem during this limiting process!

Consider the vector field $\boldsymbol{E} = \frac{\hat{\boldsymbol{r}}}{r^2}$

Q1 Can you find a scalar field such that $\boldsymbol{E} = \boldsymbol{\nabla} T$?

Q2 Is $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E} \cdot d\boldsymbol{l}$ depending on the path? Use fundamental theorem for gradients.

Q3 Show by direct calculation that $\boldsymbol{\nabla} \cdot \boldsymbol{E}$=0 for r>0. What happens at r=0?

Q4 What is the flux $\oint_S \boldsymbol{E} \cdot d\boldsymbol{a}$ on a sphere of radius R with center at origin?

Q5 Using divergence theorem calculate $\oint_S \boldsymbol{E} \cdot d\boldsymbol{a}$ on an arbitrary surface *S* which encloses he origin. Use also Q3 and Q4 results.

Q6 Show that $\boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{0}$.

Q7 Using only Q6 and Stokes' theorem show that $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E} \cdot d\boldsymbol{l}$ is independent of path.

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}. \tag{1.79}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}. \tag{1.80}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}. \tag{1.81}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}. \tag{1.82}$$

The Theory of Vector Fields

Helmholtz Theorem

If $\boldsymbol{\nabla} \cdot \boldsymbol{F} = D, \nabla \times \boldsymbol{F} = \boldsymbol{C}\ with\ \boldsymbol{\nabla} \cdot \boldsymbol{C} = 0$ with D and $\boldsymbol{C}$ given can we determine $\boldsymbol{F}$? Is $\boldsymbol{F}$ unique?

Helmholtz theorem: Yes we can determine $\boldsymbol{F}$ and $\boldsymbol{F}$ is unique provided that we require ***F***, *D* and ***C*** to go to zero at infinity.

What is the field for which $\boldsymbol{\nabla} \cdot \boldsymbol{F} = 0\ and\ \nabla \times \boldsymbol{F} = \boldsymbol{0}$ and goes to zero at infinity?