Uniform in the multiple-decrement tables

$$ {}_sp_x^{\prime(j)} = \left({}_sp_x^{(\tau)} \right)^{\frac{q_x^{(j)}}{q_x^{(\tau)}}} \quad 0 \le s \le 1 $$

$$ \frac{ln \ {}_sp_x^{\prime(j)}}{ln \ {}_sp_x^{(\tau)}} = \frac{q_x^{(j)}}{q_x^{(\tau)}} $$

Let $\mu_x^{(j)}$ and $\mu_x^{(\tau)}$ be the constant forces of decrement:

1. Relate ${}_sp_x^{\prime(j)}$ and ${}_sp_x^{(\tau)}$.

$$ {}_sp_x^{\prime(j)} = e^{-s\,\mu_x^{(j)}} $$

$$ {}_sp_x^{(\tau)} = e^{-s\,\mu_x^{(\tau)}} $$

$$ {}_sp_x^{\prime(j)} = \left({}_sp_x^{(\tau)} \right)^{\frac{\mu_x^{(j)}}{\mu_x^{(\tau)}}} $$

2. Change the ratio of μs in the exponent to a ratio of qs.

$$ q_x^{(j)} = \int_0^1 {}_tp_x^{(\tau)} \mu_x^{(j)} \, ds = \mu_x^{(j)} \int_0^1 {}_sp_x^{(\tau)} \, ds $$

$$ q_x^{(\tau)} = \int_0^1 {}_tp_x^{(\tau)} \mu_x^{(\tau)} \, ds = \mu_x^{(\tau)} \int_0^1 {}_sp_x^{(\tau)} \, ds $$

3. Dividing the first equation by the second.

$$ \frac{q_x^{(j)}}{q_x^{(\tau)}} = \frac{\mu_x^{(j)}}{\mu_x^{(\tau)}} $$

4. Replace the ratio of μs with the ratio of qs.

$$ {}_sp_x^{\prime(j)} = \left({}_sp_x^{(\tau)} \right)^{\frac{\mu_x^{(j)}}{\mu_x^{(\tau)}}} = \left({}_sp_x^{(\tau)} \right)^{\frac{q_x^{(j)}}{q_x^{(\tau)}}} $$

- $q_x^{(j)} = q_x^{(\tau)} \left(\frac{ln p_x^{\prime(j)}}{ln p_x^{(\tau)}} \right)$
- When ${}_tq_x^{(j)} = t q_x^{(j)}$ and ${}_tq_x^{(\tau)} = t q_x^{(\tau)}$ for $0 \le t \le 1$. Then,

$$ \mu_{x+t}^{(j)} = \frac{q_x^{(j)}}{{}_tp_x^{(\tau)}} = \frac{q_x^{(j)}}{1 - {}_tq_x^{(\tau)}} $$

$$ \int_0^t \mu_{x+s}^{(j)} ds = -q_x^{(j)} \frac{\ln\left(1 - {}_sq_x^{(\tau)}\right)}{q_x^{(\tau)}} \Big|_0^t = -q_x^{(j)} \frac{\ln\left(1 - {}_tq_x^{(\tau)}\right)}{q_x^{(\tau)}} = -\frac{q_x^{(j)}}{q_x^{(\tau)}} ln \ {}_tp_x^{(\tau)} $$

$$ {}_tp_x^{\prime(j)} = e^{-\int_0^t \mu_{x+s}^{(j)} ds} = \left({}_tp_x^{(\tau)} \right)^{\frac{q_x^{(j)}}{q_x^{(\tau)}}} $$

Uniform in the associated single-decrement tables

We can assume a uniform distribution for every decrement in the associated single-decrement tables.

The trick is to use the ${}_tp_x^{\prime(j)} \mu_{x+t}^{(j)} = q_x^{\prime(j)}$

- If there are two decrements, then

$$
\begin{aligned}
{}_tq_x^{(1)} &= \int_0^t {}_sp_x^{(\tau)} \mu_{x+s}^{(1)} \, ds \\
&= \int_0^t {}_sp_x^{\prime(1)} \, {}_sp_x^{\prime(2)} \mu_{x+s}^{(1)} \, ds \qquad \textit{because } {}_sp_x^{(\tau)} = {}_sp_x^{\prime(1)} \, {}_sp_x^{\prime(2)} \\
&= \int_0^t \left({}_sp_x^{\prime(1)} \mu_{x+s}^{(1)} \right) {}_sp_x^{\prime(2)} \, ds \\
&= \int_0^t q_x^{\prime(1)} \left(1 - s q_x^{\prime(2)}\right) ds \qquad \textit{because } {}_sp_x^{\prime(1)} \mu_{x+s}^{(1)} = q_x^{\prime(1)} \ \& \ {}_sp_x^{\prime(2)} = 1 - s q_x^{\prime(2)} \textit{ under UDD} \\
&= q_x^{\prime(1)} \int_0^t \left(1 - s q_x^{\prime(2)}\right) ds \\
&= q_x^{\prime(1)} \left(t - \frac{t^2 q_x^{\prime(2)}}{2}\right)
\end{aligned}
$$