Chapter 4 : Series & Sequences

In this chapter we'll be taking a look at sequences and (infinite) series. In fact, this chapter will deal almost exclusively with series. However, we also need to understand some of the basics of sequences in order to properly deal with series. We will therefore, spend a little time on sequences as well.

Series is one of those topics that many students don't find all that useful. To be honest, many students will never see series outside of their calculus class. However, series do play an important role in the field of ordinary differential equations and without series large portions of the field of partial differential equations would not be possible.

In other words, series is an important topic even if you won't ever see any of the applications. Most of the applications are beyond the scope of most Calculus courses and tend to occur in classes that many students don't take. So, as you go through this material keep in mind that these do have applications even if we won't really be covering many of them in this class.

Here is a list of topics in this chapter.

Sequences – In this section we define just what we mean by sequence in a math class and give the basic notation we will use with them. We will focus on the basic terminology, limits of sequences and convergence of sequences in this section. We will also give many of the basic facts and properties we'll need as we work with sequences.

More on Sequences – In this section we will continue examining sequences. We will determine if a sequence in an increasing sequence or a decreasing sequence and hence if it is a monotonic sequence. We will also determine a sequence is bounded below, bounded above and/or bounded.

Series – The Basics – In this section we will formally define an infinite series. We will also give many of the basic facts, properties and ways we can use to manipulate a series. We will also briefly discuss how to determine if an infinite series will converge or diverge (a more in depth discussion of this topic will occur in the next section).

Convergence/Divergence of Series – In this section we will discuss in greater detail the convergence and divergence of infinite series. We will illustrate how partial sums are used to determine if an infinite series converges or diverges. We will also give the Divergence Test for series in this section.

Special Series – In this section we will look at three series that either show up regularly or have some nice properties that we wish to discuss. We will examine Geometric Series, Telescoping Series, and Harmonic Series.

Integral Test – In this section we will discuss using the Integral Test to determine if an infinite series converges or diverges. The Integral Test can be used on a infinite series provided the terms of the series are positive and decreasing. A proof of the Integral Test is also given.

Section 4-1 : Sequences

Let's start off this section with a discussion of just what a sequence is. A sequence is nothing more than a list of numbers written in a specific order. The list may or may not have an infinite number of terms in them although we will be dealing exclusively with infinite sequences in this class. General sequence terms are denoted as follows,

$$\begin{aligned} &a_1 - \text{first term} \\ &a_2 - \text{second term} \\ &\quad\vdots \\ &a_n - n^{th} \text{ term} \\ &a_{n+1} - \left(n+1\right)^{\text{st}} \text{ term} \\ &\quad\vdots \end{aligned}$$

Because we will be dealing with infinite sequences each term in the sequence will be followed by another term as noted above. In the notation above we need to be very careful with the subscripts. The subscript of $n+1$ denotes the next term in the sequence and NOT one plus the n^{th} term! In other words,

$$a_{n+1} \ne a_n + 1$$

so be very careful when writing subscripts to make sure that the "+1" doesn't migrate out of the subscript! This is an easy mistake to make when you first start dealing with this kind of thing.

There is a variety of ways of denoting a sequence. Each of the following are equivalent ways of denoting a sequence.

$$\left\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\right\} \qquad \left\{a_n\right\} \qquad \left\{a_n\right\}_{n=1}^{\infty}$$

In the second and third notations above a_n is usually given by a formula.

A couple of notes are now in order about these notations. First, note the difference between the second and third notations above. If the starting point is not important or is implied in some way by the problem it is often not written down as we did in the third notation. Next, we used a starting point of $n=1$ in the third notation only so we could write one down. There is absolutely no reason to believe that a sequence will start at $n=1$. A sequence will start where ever it needs to start.

Let's take a look at a couple of sequences.

$$\lim_{n\to\infty}\frac{3n^2-1}{10n+5n^2}=\lim_{n\to\infty}\frac{n^2\left(3-\frac{1}{n^2}\right)}{n^2\left(\frac{10}{n}+5\right)}=\lim_{n\to\infty}\frac{3-\frac{1}{n^2}}{\frac{10}{n}+5}=\frac{3}{5}$$

So, the sequence converges and its limit is $\frac{3}{5}$.

(b) $\left\{\dfrac{\mathbf{e}^{2n}}{n}\right\}_{n=1}^{\infty}$

We will need to be careful with this one. We will need to use L'Hospital's Rule on this sequence. The problem is that L'Hospital's Rule only works on functions and not on sequences. Normally this would be a problem, but we've got Theorem 1 from above to help us out. Let's define

$$f(x)=\frac{\mathbf{e}^{2x}}{x}$$

and note that,

$$f(n)=\frac{\mathbf{e}^{2n}}{n}$$

Theorem 1 says that all we need to do is take the limit of the function.

$$\lim_{n\to\infty}\frac{\mathbf{e}^{2n}}{n}=\lim_{x\to\infty}\frac{\mathbf{e}^{2x}}{x}=\lim_{x\to\infty}\frac{2\mathbf{e}^{2x}}{1}=\infty$$

So, the sequence in this part diverges (to ∞).

More often than not we just do L'Hospital's Rule on the sequence terms without first converting to *x*'s since the work will be identical regardless of whether we use *x* or *n*. However, we really should remember that technically we can't do the derivatives while dealing with sequence terms.

(c) $\left\{\dfrac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

We will also need to be careful with this sequence. We might be tempted to just say that the limit of the sequence terms is zero (and we'd be correct). However, technically we can't take the limit of sequences whose terms alternate in sign, because we don't know how to do limits of functions that exhibit that same behavior. Also, we want to be very careful to not rely too much on intuition with these problems. As we will see in the next section, and in later sections, our intuition can lead us astray in these problems if we aren't careful.

So, let's work this one by the book. We will need to use Theorem 2 on this problem. To this we'll first need to compute,

Section 4-3 : Series - The Basics

In this section we will introduce the topic that we will be discussing for the rest of this chapter. That topic is infinite series. So just what is an infinite series? Well, let's start with a sequence $\left\{a_n\right\}_{n=1}^{\infty}$ (note the $n=1$ is for convenience, it can be anything) and define the following,

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ s_4 &= a_1 + a_2 + a_3 + a_4 \\ &\ \ \vdots \\ s_n &= a_1 + a_2 + a_3 + a_4 + \cdots + a_n = \sum_{i=1}^{n} a_i \end{aligned}$$

The s_n are called **partial sums** and notice that they will form a sequence, $\left\{s_n\right\}_{n=1}^{\infty}$. Also recall that the Σ is used to represent this summation and called a variety of names. The most common names are : **series notation**, **summation notation**, and **sigma notation**.

You should have seen this notation, at least briefly, back when you saw the definition of a definite integral in Calculus I. If you need a quick refresher on summation notation see the [review of summation notation]() in the Calculus I notes.

Now back to series. We want to take a look at the limit of the sequence of partial sums, $\left\{s_n\right\}_{n=1}^{\infty}$. To make the notation go a little easier we'll define,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \sum_{i=1}^{\infty} a_i$$

We will call $\sum_{i=1}^{\infty} a_i$ an **infinite series** and note that the series "starts" at $i=1$ because that is where our original sequence, $\left\{a_n\right\}_{n=1}^{\infty}$, started. Had our original sequence started at 2 then our infinite series would also have started at 2. The infinite series will start at the same value that the sequence of terms (as opposed to the sequence of partial sums) starts.

It is important to note that $\sum_{i=1}^{\infty} a_i$ is really nothing more than a convenient notation for $\lim_{n\to\infty} \sum_{i=1}^{n} a_i$ so we do not need to keep writing the limit down. We do, however, always need to remind ourselves that we really do have a limit there!

Section 4-4 : Convergence/Divergence of Series

In the previous section we spent some time getting familiar with series and we briefly defined convergence and divergence. Before worrying about convergence and divergence of a series we wanted to make sure that we've started to get comfortable with the notation involved in series and some of the various manipulations of series that we will, on occasion, need to be able to do.

As noted in the previous section most of what we were doing there won't be done much in this chapter. So, it is now time to start talking about the convergence and divergence of a series as this will be a topic that we'll be dealing with to one extent or another in almost all of the remaining sections of this chapter.

So, let's recap just what an infinite series is and what it means for a series to be convergent or divergent. We'll start with a sequence $\left\{a_n\right\}_{n=1}^{\infty}$ and again note that we're starting the sequence at $n=1$ only for the sake of convenience and it can, in fact, be anything.

Next, we define the partial sums of the series as,

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ s_4 &= a_1 + a_2 + a_3 + a_4 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + a_4 + \cdots + a_n = \sum_{i=1}^{n} a_i \end{aligned}$$

and these form a new sequence, $\left\{s_n\right\}_{n=1}^{\infty}$.

An infinite series, or just series here since almost every series that we'll be looking at will be an infinite series, is then the limit of the partial sums. Or,

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n$$

It is important to remember that $\sum_{i=1}^{\infty} a_i$ is really nothing more than a convenient notation for $\lim_{n\to\infty}\sum_{i=1}^{n} a_i$ so we do not need to keep writing the limit down. We do, however, always need to remind ourselves that we really do have a limit there!

If the sequence of partial sums is a convergent sequence (*i.e.* its limit exists and is finite) then the series is also called **convergent** and in this case if $\lim_{n\to\infty} s_n = s$ then, $\sum_{i=1}^{\infty} a_i = s$. Likewise, if the sequence of partial sums is a divergent sequence (*i.e.* its limit doesn't exist or is plus or minus infinity) then the series is also called **divergent**.

Therefore, a geometric series will converge if $-1 < r < 1$, which is usually written $|r| < 1$, its value is,

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^{n} = \frac{a}{1-r}$$

Note that in using this formula we'll need to make sure that we are in the correct form. In other words, if the series starts at $n = 0$ then the exponent on the *r* must be *n*. Likewise, if the series starts at $n = 1$ then the exponent on the *r* must be $n-1$.

Example 1 Determine if the following series converge or diverge. If they converge give the value of the series.

(a) $\displaystyle\sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$

Solution

(a) $\displaystyle\sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1}$

This series doesn't really look like a geometric series. However, notice that both parts of the series term are numbers raised to a power. This means that it can be put into the form of a geometric series. We will just need to decide which form is the correct form. Since the series starts at $n = 1$ we will want the exponents on the numbers to be $n-1$.

It will be fairly easy to get this into the correct form. Let's first rewrite things slightly. One of the *n*'s in the exponent has a negative in front of it and that can't be there in the geometric form. So, let's first get rid of that.

$$\sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1} = \sum_{n=1}^{\infty} 9^{-(n-2)} 4^{n+1} = \sum_{n=1}^{\infty} \frac{4^{n+1}}{9^{n-2}}$$

Now let's get the correct exponent on each of the numbers. This can be done using simple exponent properties.

$$\sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1} = \sum_{n=1}^{\infty} \frac{4^{n+1}}{9^{n-2}} = \sum_{n=1}^{\infty} \frac{4^{n-1} 4^{2}}{9^{n-1} 9^{-1}}$$

Now, rewrite the term a little.

$$\sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1} = \sum_{n=1}^{\infty} 16(9) \frac{4^{n-1}}{9^{n-1}} = \sum_{n=1}^{\infty} 144 \left(\frac{4}{9}\right)^{n-1}$$

So, this is a geometric series with $a = 144$ and $r = \frac{4}{9} < 1$. Therefore, since $|r| < 1$ we know the series will converge and its value will be,

Telescoping Series

It's now time to look at the second of the three series in this section. In this portion we are going to look at a series that is called a telescoping series. The name in this case comes from what happens with the partial sums and is best shown in an example.

Example 3 Determine if the following series converges or diverges. If it converges find its value.

$$\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}$$

Solution

We first need the partial sums for this series.

$$s_n = \sum_{i=0}^{n} \frac{1}{i^2+3i+2}$$

Now, let's notice that we can use partial fractions on the series term to get,

$$\frac{1}{i^2+3i+2} = \frac{1}{(i+2)(i+1)} = \frac{1}{i+1} - \frac{1}{i+2}$$

We'll leave the details of the partial fractions to you. By now you should be fairly adept at this since we spent a fair amount of time doing partial fractions back in the Integration Techniques chapter. If you need a refresher you should go back and review that section.

So, what does this do for us? Well, let's start writing out the terms of the general partial sum for this series using the partial fraction form.

$$\begin{aligned} s_n &= \sum_{i=0}^{n} \left(\frac{1}{i+1} - \frac{1}{i+2} \right) \\ &= \left(\frac{1}{1} - \cancel{\frac{1}{2}} \right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} \right) + \cdots + \left(\cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}} \right) + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2} \right) \\ &= 1 - \frac{1}{n+2} \end{aligned}$$

Notice that every term except the first and last term canceled out. This is the origin of the name **telescoping series**.

This also means that we can determine the convergence of this series by taking the limit of the partial sums.

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+2} \right) = 1$$

The sequence of partial sums is convergent and so the series is convergent and has a value of

$$\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2} = 1$$

$$
\begin{aligned}
A &\approx \left(\frac{1}{1}\right)(1)+\left(\frac{1}{2}\right)(1)+\left(\frac{1}{3}\right)(1)+\left(\frac{1}{4}\right)(1)+\left(\frac{1}{5}\right)(1)+\cdots \\
&= \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots \\
&= \sum_{n=1}^{\infty}\frac{1}{n}
\end{aligned}
$$

Now note a couple of things about this approximation. First, each of the rectangles overestimates the actual area and secondly the formula for the area is exactly the harmonic series!

Putting these two facts together gives the following,

$$A \approx \sum_{n=1}^{\infty}\frac{1}{n} > \int_{1}^{\infty}\frac{1}{x}\,dx = \infty$$

Notice that this tells us that we must have,

$$\sum_{n=1}^{\infty}\frac{1}{n} > \infty \qquad\qquad \Rightarrow \qquad\qquad \sum_{n=1}^{\infty}\frac{1}{n} = \infty$$

Since we can't really be larger than infinity the harmonic series must also be infinite in value. In other words, the harmonic series is in fact divergent.

So, we've managed to relate a series to an improper integral that we could compute and it turns out that the improper integral and the series have exactly the same convergence.

Let's see if this will also be true for a series that converges. When discussing the Divergence Test we made the claim that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

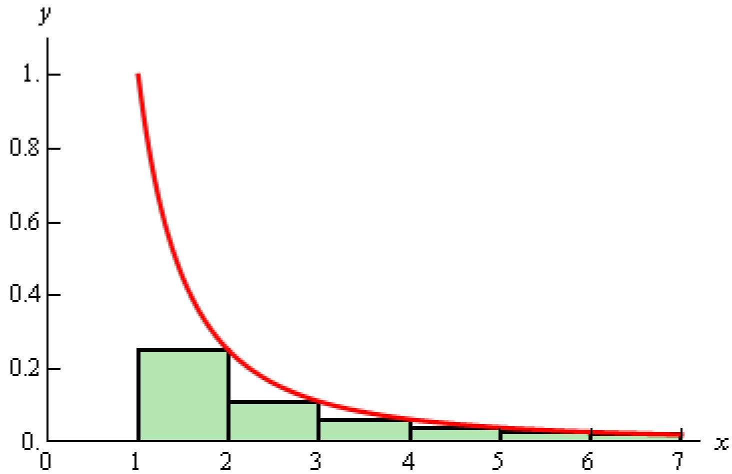
converges. Let's see if we can do something similar to the above process to prove this.

We will try to relate this to the area under $f(x)=\frac{1}{x^2}$ is on the interval $[1,\infty)$. Again, from the Improper Integral section we know that,

$$\int_{1}^{\infty}\frac{1}{x^2}\,dx = 1$$

and so this integral converges.

We will once again try to estimate the area under this curve. We will do this in an almost identical manner as the previous part with the exception that instead of using the left end points for the height of our rectangles we will use the right end points. Here is a sketch of this case,



In this case the area estimation is,

$$A \approx \left(\frac{1}{2^2}\right)(1) + \left(\frac{1}{3^2}\right)(1) + \left(\frac{1}{4^2}\right)(1) + \left(\frac{1}{5^2}\right)(1) + \cdots$$
$$= \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

This time, unlike the first case, the area will be an underestimation of the actual area and the estimation is not quite the series that we are working with. Notice however that the only difference is that we're missing the first term. This means we can do the following,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \underbrace{\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots}_{\text{Area Estimation}} < 1 + \int_1^{\infty} \frac{1}{x^2}\,dx = 1 + 1 = 2$$

Or, putting all this together we see that,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$$

With the harmonic series this was all that we needed to say that the series was divergent. With this series however, this isn't quite enough. For instance, $-\infty < 2$, and if the series did have a value of $-\infty$ then it would be divergent (when we want convergent). So, let's do a little more work.

First, let's notice that all the series terms are positive (that's important) and that the partial sums are,

$$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$$

Because the terms are all positive we know that the partial sums must be an increasing sequence. In other words,

$$s_n = \sum_{i=1}^{n} \frac{1}{i^2} < \sum_{i=1}^{n+1} \frac{1}{i^2} = s_{n+1}$$

Recall that we had a similar test for improper integrals back when we were looking at integration techniques. So, if you could use the comparison test for improper integrals you can use the comparison test for series as they are pretty much the same idea.

Note as well that the requirement that $a_n, b_n \geq 0$ and $a_n \leq b_n$ really only need to be true eventually. In other words, if a couple of the first terms are negative or $a_n \not\leq b_n$ for a couple of the first few terms we're okay. As long as we eventually reach a point where $a_n, b_n \geq 0$ and $a_n \leq b_n$ for all sufficiently large *n* the test will work.

To see why this is true let's suppose that the series start at $n = k$ and that the conditions of the test are only true for for $n \geq N+1$ and for $k \leq n \leq N$ at least one of the conditions is not true. If we then look at $\sum a_n$ (the same thing could be done for $\sum b_n$) we get,

$$\sum_{n=k}^{\infty} a_n = \sum_{n=k}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

The first series is nothing more than a finite sum (no matter how large *N* is) of finite terms and so will be finite. So, the original series will be convergent/divergent only if the second infinite series on the right is convergent/divergent and the test can be done on the second series as it satisfies the conditions of the test.

Let's take a look at some examples.

Example 1 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n}{n^2 - \cos^2(n)}$$

Solution

Since the cosine term in the denominator doesn't get too large we can assume that the series terms will behave like,

$$\frac{n}{n^2} = \frac{1}{n}$$

which, as a series, will diverge. So, from this we can guess that the series will probably diverge and so we'll need to find a smaller series that will also diverge.

Recall that from the comparison test with improper integrals that we determined that we can make a fraction smaller by either making the numerator smaller or the denominator larger. In this case the two terms in the denominator are both positive. So, if we drop the cosine term we will in fact be making the denominator larger since we will no longer be subtracting off a positive quantity. Therefore,

$$\frac{n}{n^2 - \cos^2(n)} > \frac{n}{n^2} = \frac{1}{n}$$

Then, since

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges (it's harmonic or the *p*-series test) by the Comparison Test our original series must also diverge.

Example 2 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\mathbf{e}^{-n}}{n+\cos^2(n)}$$

Solution

This example looks somewhat similar to the first one but we are going to have to be careful with it as there are some significant differences.

First, as with the first example the cosine term in the denominator will not get very large and so it won't affect the behavior of the terms in any meaningful way. Therefore, the temptation at this point is to focus in on the *n* in the denominator and think that because it is just an *n* the series will diverge.

That would be correct if we didn't have much going on in the numerator. In this example, however, we also have an exponential in the numerator that is going to zero very fast. In fact, it is going to zero so fast that it will, in all likelihood, force the series to converge.

So, let's guess that this series will converge and we'll need to find a larger series that will also converge.

First, because we are adding two positive numbers in the denominator we can drop the cosine term from the denominator. This will, in turn, make the denominator smaller and so the term will get larger or,

$$\frac{\mathbf{e}^{-n}}{n+\cos^2(n)} \le \frac{\mathbf{e}^{-n}}{n}$$

Next, we know that $n \ge 1$ and so if we replace the *n* in the denominator with its smallest possible value (*i.e.* 1) the term will again get larger. Doing this gives,

$$\frac{\mathbf{e}^{-n}}{n+\cos^2(n)} \le \frac{\mathbf{e}^{-n}}{n} \le \frac{\mathbf{e}^{-n}}{1} = \mathbf{e}^{-n}$$

We can't do much more, in a way that is useful anyway, to make this larger so let's see if we can determine if,

$$\sum_{n=1}^{\infty} \mathbf{e}^{-n}$$

converges or diverges.

We can notice that $f(x) = \mathbf{e}^{-x}$ is always positive and it is also decreasing (you can verify that correct?) and so we can use the Integral Test on this series. Doing this gives,

$$\int_1^\infty \mathbf{e}^{-x}\,dx = \lim_{t\to\infty}\int_1^t \mathbf{e}^{-x}\,dx = \lim_{t\to\infty}\left(-\mathbf{e}^{-x}\right)\Big|_1^t = \lim_{t\to\infty}\left(-\mathbf{e}^{-t}+\mathbf{e}^{-1}\right) = \mathbf{e}^{-1}$$

Okay, we now know that the integral is convergent and so the series $\sum_{n=1}^{\infty}\mathbf{e}^{-n}$ must also be convergent.

Therefore, because $\sum_{n=1}^{\infty}\mathbf{e}^{-n}$ is larger than the original series we know that the original series must also converge.

With each of the previous examples we saw that we can't always just focus in on the denominator when making a guess about the convergence of a series. Sometimes there is something going on in the numerator that will change the convergence of a series from what the denominator tells us should be happening.

We also saw in the previous example that, unlike most of the examples of the comparison test that we've done (or will do) both in this section and in the Comparison Test for Improper Integrals, that it won't always be the denominator that is driving the convergence or divergence. Sometimes it is the numerator that will determine if something will converge or diverge so do not get too locked into only looking at the denominator.

One of the more common mistakes is to just focus in on the denominator and make a guess based just on that. If we'd done that with both of the previous examples we would have guessed wrong so be careful.

Let's work another example of the comparison test before we move on to a different topic.

Example 3 Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{n^2+2}{n^4+5}$$

Solution

In this case the "+2" and the "+5" don't really add anything to the series and so the series terms should behave pretty much like

$$\frac{n^2}{n^4} = \frac{1}{n^2}$$

which will converge as a series. Therefore, we can guess that the original series will converge and we will need to find a larger series which also converges.

This means that we'll either have to make the numerator larger or the denominator smaller. We can make the denominator smaller by dropping the "+5". Doing this gives,

$$\frac{n^2+2}{n^4+5} < \frac{n^2+2}{n^4}$$

bounded sequence is also convergent and so $\left\{s_n\right\}_{n=1}^{\infty}$ is a convergent sequence and so $\sum_{n=1}^{\infty} a_n$ is convergent.

Next, let's assume that $\sum_{n=1}^{\infty} a_n$ is divergent. Because $a_n \geq 0$ we then know that we must have $s_n \to \infty$ as $n \to \infty$. However, we also know that for all *n* we have $s_n \leq t_n$ and therefore we also know that $t_n \to \infty$ as $n \to \infty$.

So, $\left\{t_n\right\}_{n=1}^{\infty}$ is a divergent sequence and so $\sum_{n=1}^{\infty} b_n$ is divergent.

Proof of Limit Comparison Test

Because $0 < c < \infty$ we can find two positive and finite numbers, *m* and *M*, such that $m < c < M$. Now, because $c = \lim_{n \to \infty} \frac{a_n}{b_n}$ we know that for large enough *n* the quotient $\frac{a_n}{b_n}$ must be close to *c* and so there must be a positive integer *N* such that if $n > N$ we also have,

$$m < \frac{a_n}{b_n} < M$$

Multiplying through by b_n gives,

$$mb_n < a_n < Mb_n$$

provided $n > N$.

Now, if $\sum b_n$ diverges then so does $\sum mb_n$ and so since $mb_n < a_n$ for all sufficiently large *n* by the Comparison Test $\sum a_n$ also diverges.

Likewise, if $\sum b_n$ converges then so does $\sum Mb_n$ and since $a_n < Mb_n$ for all sufficiently large *n* by the Comparison Test $\sum a_n$ also converges.

Now, the second part of this clearly is going to 1 as $n \to \infty$ while the first part just alternates between 1 and -1. So, as $n \to \infty$ the terms are alternating between positive and negative values that are getting closer and closer to 1 and -1 respectively.

In order for limits to exist we know that the terms need to settle down to a single number and since these clearly don't this limit doesn't exist and so by the Divergence Test this series diverges.

Example 3 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4}$$

Solution

Notice that in this case the exponent on the "-1" isn't *n* or $n+1$. That won't change how the test works however so we won't worry about that. In this case we have,

$$b_n = \frac{\sqrt{n}}{n+4}$$

so let's check the conditions.

The first is easy enough to check.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+4} = 0$$

The second condition requires some work however. It is not immediately clear that these terms will decrease. Increasing *n* to $n+1$ will increase both the numerator and the denominator. Increasing the numerator says the term should also increase while increasing the denominator says that the term should decrease. Since it's not clear which of these will win out we will need to resort to Calculus I techniques to show that the terms decrease.

Let's start with the following function and its derivative.

$$f(x) = \frac{\sqrt{x}}{x+4} \qquad\qquad f'(x) = \frac{4-x}{2\sqrt{x}\,(x+4)^2}$$

Now, there are two critical points for this function, $x=0$, and $x=4$. Note that $x=-4$ is not a critical point because the function is not defined at $x=-4$. The first is outside the bound of our series so we won't need to worry about that one. Using the test points,

$$f'(1) = \frac{3}{50} \qquad\qquad f'(5) = -\frac{\sqrt{5}}{810}$$

and so we can see that the function in increasing on $0 \le x \le 4$ and decreasing on $x \ge 4$. Therefore, since $f(n) = b_n$ we know as well that the b_n are also increasing on $0 \le n \le 4$ and decreasing on $n \ge 4$.

Example 3 Determine if the following series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{n^2}{(2n-1)!}$$

Solution

In this case be careful in dealing with the factorials.

$$\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{(n+1)^2}{(2(n+1)-1)!}\frac{(2n-1)!}{n^2}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)^2}{(2n+1)!}\frac{(2n-1)!}{n^2}\right| \\
&= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n)(2n-1)!}\frac{(2n-1)!}{n^2} \\
&= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n)(n^2)} \\
&= 0 < 1
\end{aligned}$$

So, by the Ratio Test this series converges absolutely and so converges.

Example 4 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{9^n}{(-2)^{n+1} n}$$

Solution

Do not mistake this for a geometric series. The n in the denominator means that this isn't a geometric series. So, let's compute the limit.

$$\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{9^{n+1}}{(-2)^{n+2}(n+1)}\frac{(-2)^{n+1}\,n}{9^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{9n}{(-2)(n+1)}\right| \\
&= \frac{9}{2}\lim_{n\to\infty}\frac{n}{n+1} \\
&= \frac{9}{2} > 1
\end{aligned}$$

Therefore, by the Ratio Test this series is divergent.

Section 4-11 : Root Test

This is the last test for series convergence that we're going to be looking at. As with the Ratio Test this test will also tell whether a series is absolutely convergent or not rather than simple convergence.

Root Test

Suppose that we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty} \sqrt[n]{\left|a_n\right|} = \lim_{n\to\infty} \left|a_n\right|^{\frac{1}{n}}$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

A proof of this test is at the end of the section.

As with the ratio test, if we get $L = 1$ the root test will tell us nothing and we'll need to use another test to determine the convergence of the series. Also note that, generally for the series we'll be dealing with in this class, if $L = 1$ in the Ratio Test then the Root Test will also give $L = 1$.

We will also need the following fact in some of these problems.

Fact

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

Let's take a look at a couple of examples.

Example 1 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$$

Solution

There really isn't much to these problems other than computing the limit and then using the root test. Here is the limit for this problem.

$$L = \lim_{n\to\infty} \left|\frac{n^n}{3^{1+2n}}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3^{\frac{1}{n}+2}} = \frac{\infty}{3^2} = \infty > 1$$

So, by the Root Test this series is divergent.

The series on the left is in the standard form and so we can compute that directly. The first series on the right has a finite number of terms and so can be computed exactly and the second series on the right is the one that we'd like to have the value for. Doing the work gives,

$$\begin{aligned}\sum_{n=16}^{\infty}\left(\frac{1}{2}\right)^n &= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n - \sum_{n=0}^{15}\left(\frac{1}{2}\right)^n \\ &= \frac{1}{1-\left(\frac{1}{2}\right)} - 1.999969482 \\ &= 0.000030518\end{aligned}$$

So, according to this if we use

$$s \approx 1.383062486$$

as an estimate of the actual value we will be off from the exact value by no more than 0.000030518 and that's not too bad.

In this case it can be shown that

$$\sum_{n=0}^{\infty}\frac{2^n}{4^n+1} = 1.383093004$$

and so we can see that the actual error in our estimation is,

$$\text{Error} = \text{Actual} - \text{Estimate} = 1.383093004 - 1.383062486 = 0.000030518$$

Note that in this case the estimate of the error is actually fairly close (and in fact exactly the same) as the actual error. This will not always happen and so we shouldn't expect that to happen in all cases. The error estimate above is simply the upper bound on the error and the actual error will often be less than this value.

Before moving on to the final part of this section let's again note that we will only be able to determine how good the estimate is using the comparison test if we can easily get our hands on the remainder of the second term. The reality is that we won't always be able to do this.

Alternating Series Test

Both of the methods that we've looked at so far have required the series to contain only positive terms. If we allow series to have negative terms in it the process is usually more difficult. However, with that said there is one case where it isn't too bad. That is the case of an alternating series.

Once again we will start off with a convergent series $\sum a_n = \sum (-1)^n b_n$ which in this case happens to be an alternating series that satisfies the conditions of the alternating series test, so we know that $b_n \geq 0$ and is decreasing for all *n*. Also note that we could have any power on the "-1" we just used *n* for the sake of convenience. We want to know how good of an estimation of the actual series value will the partial sum, s_n, be. As with the prior cases we know that the remainder, R_n, will be the error in the estimation and so if we can get a handle on that we'll know approximately how good the estimation is.

In this case we've used the ratio test to show that $\sum a_n$ is convergent. To do this we computed

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

and found that $L < 1$.

As with the previous cases we are going to use the remainder, R_n, to determine how good of an estimation of the actual value the partial sum, s_n, is.

To get an estimate of the remainder let's first define the following sequence,

$$r_n = \frac{a_{n+1}}{a_n}$$

We now have two possible cases.

1. If $\{r_n\}$ is a decreasing sequence and $r_{n+1} < 1$ then,
$$R_n \le \frac{a_{n+1}}{1 - r_{n+1}}$$
2. If $\{r_n\}$ is an increasing sequence then,
$$R_n \le \frac{a_{n+1}}{1 - L}$$

Proof

Both parts will need the following work so we'll do it first. We'll start with the remainder.

$$\begin{aligned} R_n = \sum_{i=n+1}^{\infty} a_i &= a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4} + \cdots \\ &= a_{n+1}\left(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+1}} + \frac{a_{n+4}}{a_{n+1}} + \cdots\right) \end{aligned}$$

Next, we need to do a little work on a couple of these terms.

$$\begin{aligned} R_n &= a_{n+1}\left(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+1}}\frac{a_{n+2}}{a_{n+2}} + \frac{a_{n+4}}{a_{n+1}}\frac{a_{n+2}}{a_{n+2}}\frac{a_{n+3}}{a_{n+3}} + \cdots\right) \\ &= a_{n+1}\left(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+2}}{a_{n+1}}\frac{a_{n+3}}{a_{n+2}} + \frac{a_{n+2}}{a_{n+1}}\frac{a_{n+3}}{a_{n+2}}\frac{a_{n+4}}{a_{n+3}} + \cdots\right) \end{aligned}$$

Now use the definition of r_n to write this as, Now let's do a little simplification on the series.

$$\begin{aligned} g(x) &= \frac{1}{5}\sum_{n=0}^{\infty}\frac{x^n}{5^n} \\ &= \sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}} \end{aligned}$$

The interval of convergence for this series is,

$$\left|\frac{x}{5}\right| < 1 \qquad \Rightarrow \qquad \frac{1}{5}|x| < 1 \qquad \Rightarrow \qquad |x| < 5$$

Okay, this was the work for the power series representation for $g(x)$ let's now find a power series representation for the original function. All we need to do for this is to multiply the power series representation for $g(x)$ by *x* and we'll have it.

$$\begin{aligned} f(x) &= x\,\frac{1}{5-x} \\ &= x\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}} \\ &= \sum_{n=0}^{\infty}\frac{x^{n+1}}{5^{n+1}} \end{aligned}$$

The interval of convergence doesn't change and so it will be $|x| < 5$.

So, hopefully we now have an idea on how to find the power series representation for some functions. Admittedly all of the functions could be related back to (2) but it's a start.

We now need to look at some further manipulation of power series that we will need to do on occasion. We need to discuss differentiation and integration of power series.

Let's start with differentiation of the power series,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

Now, we know that if we differentiate a finite sum of terms all we need to do is differentiate each of the terms and then add them back up. With infinite sums there are some subtleties involved that we need to be careful with but are somewhat beyond the scope of this course.

Nicely enough for us however, it is known that if the power series representation of $f(x)$ has a radius of convergence of $R > 0$ then the term by term differentiation of the power series will also have a radius of convergence of *R* and (more importantly) will in fact be the power series representation of $f'(x)$ provided we stay within the radius of convergence.

$$c_2 = \frac{f''(a)}{2}$$

Using the third derivative gives,

$$f'''(x) = 3(2)c_3 + 4(3)(2)c_4(x-a) + \cdots$$

$$f'''(a) = 3(2)c_3 \qquad \Rightarrow \qquad c_3 = \frac{f'''(a)}{3(2)}$$

Using the fourth derivative gives,

$$f^{(4)}(x) = 4(3)(2)c_4 + 5(4)(3)(2)c_5(x-a)\cdots$$

$$f^{(4)}(a) = 4(3)(2)c_4 \qquad \Rightarrow \qquad c_4 = \frac{f^{(4)}(a)}{4(3)(2)}$$

Hopefully by this time you've seen the pattern here. It looks like, in general, we've got the following formula for the coefficients.

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This even works for $n = 0$ if you recall that $0! = 1$ and define $f^{(0)}(x) = f(x)$.

So, provided a power series representation for the function $f(x)$ about $x = a$ exists the **Taylor Series for $f(x)$ about $x = a$** is,

Taylor Series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

If we use $a = 0$, so we are talking about the Taylor Series about $x = 0$, we call the series a **Maclaurin Series** for $f(x)$ or,

Maclaurin Series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \end{aligned}$$

more time since it makes some of the work easier. This will be the final Taylor Series for exponentials in this section.

Example 4 Find the Taylor Series for $f\left(x\right) = \mathbf{e}^{-x}$ about $x = -4$.

Solution

Finding a general formula for $f^{(n)}\left(-4\right)$ is fairly simple.

$$f^{(n)}\left(x\right) = \left(-1\right)^n \mathbf{e}^{-x} \qquad\qquad f^{(n)}\left(-4\right) = \left(-1\right)^n \mathbf{e}^{4}$$

The Taylor Series is then,

$$\mathbf{e}^{-x} = \sum_{n=0}^{\infty} \frac{\left(-1\right)^n \mathbf{e}^{4}}{n!}\left(x+4\right)^n$$

Okay, we now need to work some examples that don't involve the exponential function since these will tend to require a little more work.

Example 5 Find the Taylor Series for $f\left(x\right) = \cos\left(x\right)$ about $x = 0$.

Solution

First, we'll need to take some derivatives of the function and evaluate them at $x = 0$.

$$\begin{aligned} f^{(0)}\left(x\right) &= \cos x & \qquad f^{(0)}\left(0\right) &= 1 \\ f^{(1)}\left(x\right) &= -\sin x & f^{(1)}\left(0\right) &= 0 \\ f^{(2)}\left(x\right) &= -\cos x & f^{(2)}\left(0\right) &= -1 \\ f^{(3)}\left(x\right) &= \sin x & f^{(3)}\left(0\right) &= 0 \\ f^{(4)}\left(x\right) &= \cos x & f^{(4)}\left(0\right) &= 1 \\ f^{(5)}\left(x\right) &= -\sin x & f^{(5)}\left(0\right) &= 0 \\ f^{(6)}\left(x\right) &= -\cos x & f^{(6)}\left(0\right) &= -1 \\ &\vdots & &\vdots \end{aligned}$$

In this example, unlike the previous ones, there is not an easy formula for either the general derivative or the evaluation of the derivative. However, there is a clear pattern to the evaluations. So, let's plug what we've got into the Taylor series and see what we get,

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$
$$= \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

In this case we only get terms that have an odd exponent on *x* and as with the last problem once we ignore the zero terms there is a clear pattern and formula. So renumbering the terms as we did in the previous example we get the following Taylor Series.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

We really need to work another example or two in which $f(x)$ isn't about $x = 0$.

Example 7 Find the Taylor Series for $f(x) = \ln(x)$ about $x = 2$.

Solution
Here are the first few derivatives and the evaluations.

$$
\begin{aligned}
f^{(0)}(x) &= \ln(x) & f^{(0)}(2) &= \ln 2 \\
f^{(1)}(x) &= \frac{1}{x} & f^{(1)}(2) &= \frac{1}{2} \\
f^{(2)}(x) &= -\frac{1}{x^2} & f^{(2)}(2) &= -\frac{1}{2^2} \\
f^{(3)}(x) &= \frac{2}{x^3} & f^{(3)}(2) &= \frac{2}{2^3} \\
f^{(4)}(x) &= -\frac{2(3)}{x^4} & f^{(4)}(2) &= -\frac{2(3)}{2^4} \\
f^{(5)}(x) &= \frac{2(3)(4)}{x^5} & f^{(5)}(2) &= \frac{2(3)(4)}{2^5} \\
&\vdots & &\vdots \\
f^{(n)}(x) &= \frac{(-1)^{n+1}(n-1)!}{x^n} & f^{(n)}(2) &= \frac{(-1)^{n+1}(n-1)!}{2^n} \qquad n = 1, 2, 3, \ldots
\end{aligned}
$$

Note that while we got a general formula here it doesn't work for $n = 0$. This will happen on occasion so don't worry about it when it does.

In order to plug this into the Taylor Series formula we'll need to strip out the $n = 0$ term first.