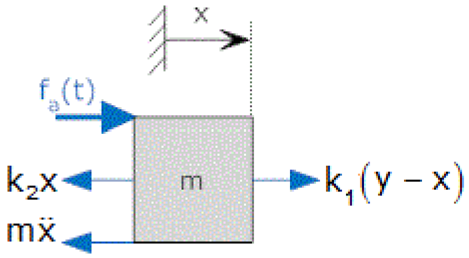
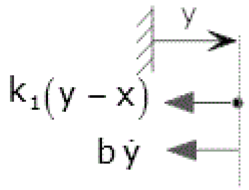
Freebody Diagram **Equation**



$$m\cdot\ddot{x}+k_1\cdot x+k_2x-k_1\cdot y=f_a$$



$$b\cdot\dot{y}+k_1y-k_1\cdot x=0$$

There are three energy storage elements, so we expect three state equations. The energy storage elements are the spring, k_2, the mass, m, and the spring, k_1. Therefore we choose as our state variables x (the energy in spring k_2 is $\frac{1}{2}k_2x^2$), the velocity at x (the energy in the mass m is $\frac{1}{2}mv^2$, where v is the first derivative of x), and y (the energy in spring k_1 is $\frac{1}{2}k_1(y-x)^2$, so we could pick y-x as a state variable, but we'll just use y (since x is already a state variable; recall that the choice of state variables is not unique). Our state variables become:

$$q_1=x$$
$$q_2=\dot{x}$$
$$q_3=y$$

Now we want equations for their derivatives. The equations of motion from the free body diagrams yield

$$\dot{q}_1=\dot{x}=q_2$$
$$\dot{q}_2=\ddot{x}=\frac{1}{m}\left(f_a-k_1x-k_2x+k_1y\right)$$
$$=\frac{1}{m}\left(f_a-k_1q_2-k_2q_1+k_1q_3\right)$$
$$\dot{q}_3=\dot{y}=\frac{k_1}{b}\left(x-y\right)=\frac{k_1}{b}\left(q_1-q_3\right)$$

or

$$\dot{\mathbf{q}}=\mathbf{A}\mathbf{q}+\mathbf{B}u \qquad \mathbf{A}=\begin{bmatrix}0 & 1 & 0\\ -\frac{k_1+k_2}{m} & 0 & \frac{k_1}{m}\\ \frac{k_1}{b} & 0 & -\frac{k_1}{b}\end{bmatrix} \qquad \mathbf{B}=\begin{bmatrix}0\\ \frac{1}{m}\\ 0\end{bmatrix}$$

$$y=\mathbf{C}\mathbf{q}+Du \qquad \mathbf{C}=\begin{bmatrix}0 & 0 & 1\end{bmatrix} \qquad D=0$$

$$\begin{aligned} q_1 &= y \\ q_2 &= \dot{y} \\ q_3 &= \ddot{y} \end{aligned}$$

Taking the derivatives we can develop our state space model

$$\begin{aligned} \dot{q}_1 &= q_2 = \dot{y} \\ \dot{q}_2 &= q_3 = \ddot{y} \\ \dot{q}_3 &= \dddot{y} = -a_3 y - a_2 \dot{y} - a_1 \ddot{y} + b_0 u \\ &= -a_3 q_1 - a_2 q_2 - a_1 q_3 + b_0 u \end{aligned}$$

$$\dot{\mathbf{q}} = \mathbf{Aq} + \mathbf{B}u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \mathbf{q} + \begin{bmatrix} 0 \\ 0 \\ b_0 \end{bmatrix} u$$

$$y = \mathbf{Cq} + Du = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{q} + 0 \cdot u$$

*Note: For an nth order system the matrices generalize in the obvious way (**A** has ones above the main diagonal and the differential equation constants for the last row, **B** is all zeros with b_0 in the bottom row, **C** is zero except for the leftmost element which is one, and D is zero)*

Repeat Starting from Transfer Function

Consider the transfer function with a constant numerator (note: this is the same system as in the preceding example). We'll use a third order equation, thought it generalizes to n^{th} order in the obvious way.

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$\left(s^3 + a_1 s^2 + a_2 s + a_3\right) Y(s) = b_0 U(s)$$

For such systems (no derivatives of the input) we can choose as our n state variables the variable y and its first n-1 derivatives (in this case the first two derivatives)

$$\begin{aligned} q_1(t) &= y(t) & Q_1(s) &= Y(s) \\ q_2(t) &= \dot{y}(t) & Q_2(s) &= sY(s) \\ q_3(t) &= \ddot{y}(t) & Q_3(s) &= s^2 Y(s) \end{aligned}$$

Taking the derivatives we can develop our state space model (which is exactly the same as when we started from the differential equation).

Key Concept: Transfer function to State Space (OCF)

For a general n[th] order transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

the observable canonical state space model form is

$$\dot{\mathbf{q}} = \mathbf{Aq} + \mathbf{B}u; \quad \mathbf{A} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ -a_{n-1} & 0 & \vdots & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ \vdots \\ b_{n-1} - a_{n-1} b_0 \\ b_n - a_n b_0 \end{bmatrix}$$

$$y = \mathbf{Cq} + Du \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \qquad D = b_0$$

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C\Phi}(s)\mathbf{B} + D = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

$$= \frac{\mathbf{C}\ \text{Adj}\ SI - A\ \mathbf{B}\ +\ SI - A\ D}{SI - A}$$

$SI - A$ is also known as characteristic equation when equated to zero.

MATLab Code

Transfer Function to State Space (tf2ss)

$$\frac{Y(s)}{U(s)} = \frac{s}{s^3 + 14s^2 + 56s + 160}$$

```
num=[1 0];
den=[1 14 56 160];
[A,B,C,D]=tf2ss(num,den)
```

A =

-14	-56	-160
1	0	0
0	1	0

$Y = \begin{bmatrix} 1 & 0 \\ & 1 \end{bmatrix} x$

(a) Find the complete solution for y(t) when

$U(t)=1(t)$, $x_1(0)=1$ and $x_2(0)=0$

(b) Determine the transfer function

(c) Draw a block diagram representing the system **[9+4+3]**

16.(a) Derive an expression for the transfer function of a pump controlled hydraulic system. State the assumption made. **[8]**

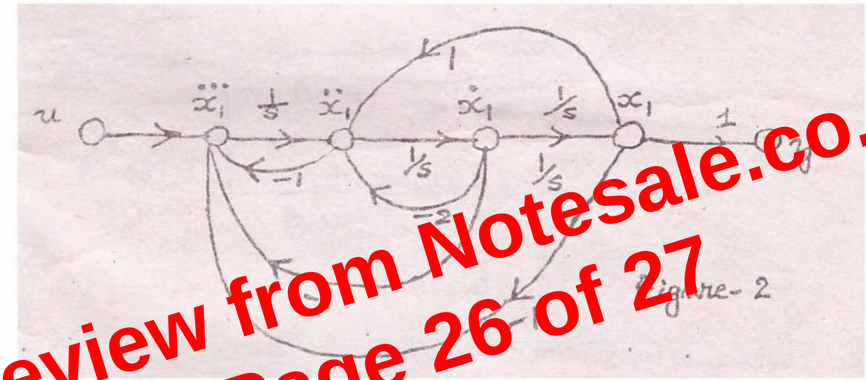
(b) Simulate a pneumatic PID controller and obtain its linearized transfer function. **[8]**

17. Describe the constructional features of a rate gyro, explain its principle of operation and obtain its transfer function. **[8]**

18. (a) Explain how poles of a closed loop control system can be placed at the desired points on the s plane. **[4]**

(b) Explain how diagonalisation of a system matrix a helps in the study of controllability of control systems. **[4]**

19. Construct the state space model of the system whose signal flow graph is shown in fig 2. **[7]**



20. (a) Define state of a system, state variables, state space and state vector. What are the advantages of state space analysis? **[5]**

(b) A two input two output linear dynamic system is governed by

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} X(t) + \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} R(t)$$

$$Y(t) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} X(t)$$

i) Find out the transfer function matrix. **[5]**

ii) Assuming $X(0) = 0$ find the output response Y(t) if **[5]**

$$R(t) = \begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix} \text{ for } t \geq 0$$

21.(a) A system is described by **[8]**

$$\dot{X}(t) = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix} X(t)$$

Diagonalise the above system making use of suitable transformation X=PZ

(b) Show how can you compute e^{At} using results of (a) **[7]**

22. Define controllability and observability and of control systems. **[4]**

23. A feed back system has a closed loop transfer function: