1 Viewing a Matrix – 4 Ways

A matrix ($m \times n$) can be seen as 1 matrix, mn numbers, n columns and m rows.

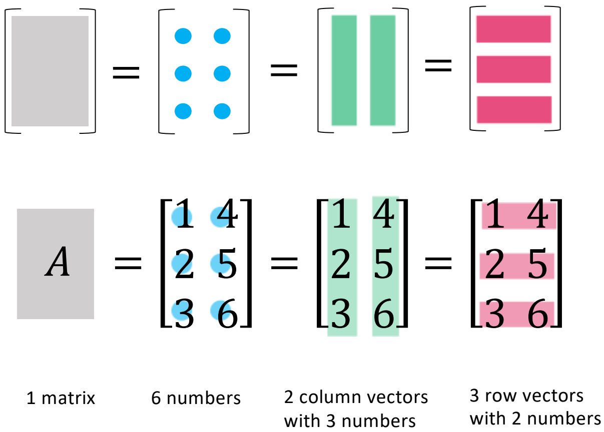


Figure 1: Viewing a Matrix in 4 Ways

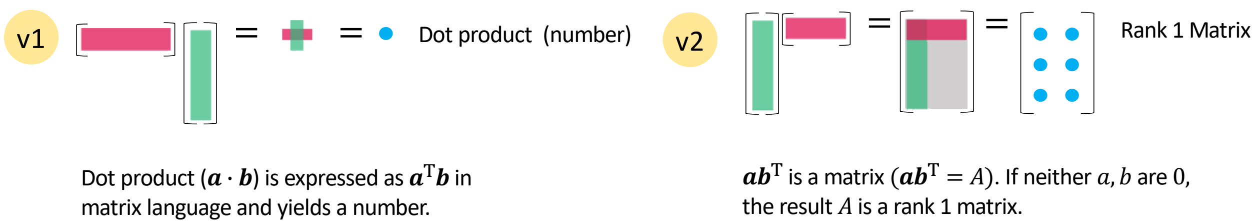
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} | & | \\ \boldsymbol{a_1} & \boldsymbol{a_2} \\ | & | \end{bmatrix} = \begin{bmatrix} -\boldsymbol{a}_1^* - \\ -\boldsymbol{a}_2^* - \\ -\boldsymbol{a}_3^* - \end{bmatrix}$$

Here, the column vectors are in bold as $\boldsymbol{a_1}$. Row vectors include $*$ as in $\boldsymbol{a}_1^*$. Transposed vectors and matrices are indicated by T as in $\boldsymbol{a}^{\mathrm{T}}$ and A^{T}.

2 Vector times Vector – 2 Ways

Hereafter I point to specific sections of "Linear Algebra for Everyone" and present graphics which illustrate the concepts with short names in colored circles.

- Sec. 1.1 (p.2) Linear combination and dot products
- Sec. 1.3 (p.25) Matrix of Rank One
- Sec. 1.4 (p.29) Row way and column way



$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 + 2x_2 + 3x_3 \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} x & y \\ 2x & 2y \\ 3x & 3y \end{bmatrix}$$

Figure 2: Vector times Vector - (v1), (v2)

(v1) is a elementary operation of two vectors, but (v2) multiplies the column to the row and produce a rank 1 matrix. Knowing this outer product (v2) is the key for the later sections.

6.1 $\boldsymbol{A = CR}$

- Sec.1.4 Matrix Multiplication and $\boldsymbol{A = CR}$ (p.29)

All general rectangular matrices A have the same row rank as the column rank. This factorization is the most intuitive way to understand this theorem. C consists of independent columns of A, and R is the row reduced echelon form of A. $A = CR$ reduces to r independent columns in C times r independent rows in R.

$$A = CR$$
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Procedure: Look at the columns of A from left to right. Keep independent ones, discard dependent ones which can be created by the former columns. The column 1 and the column 2 survive, and the column 3 is discarded because it is expressed as a sum of the former two columns. To rebuild A by the independent columns 1, 2, you find a row echelon form R appearing in the right.

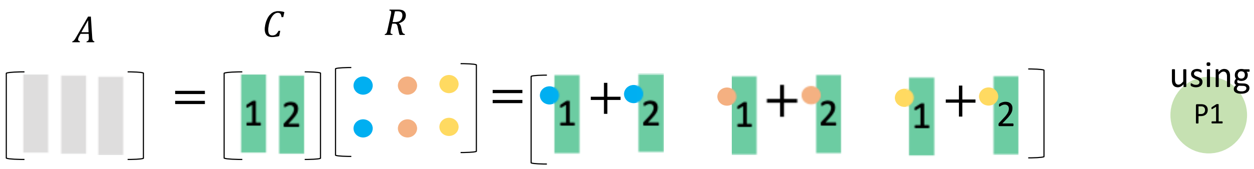


Figure 12: Column Rank in CR

Now you see the column rank is two because there are only two independent columns in C and all the columns of A are linear combinations of the two columns of C.

Figure 13: Row Rank in CR

And you see the row rank is two because there are only two independent rows in R and all the rows of A are linear combinations of the two rows of R.