Henceforth, we proceed with the integral dependent on s.

$$\frac{1}{n!}\int_0^1 (x-c)(x-(c+s(x-c)))^n\, f^{(n+1)}(c+s(x-c))\, ds =$$

$$= \frac{1}{n!}\int_0^1 (x-c)(x-c-sx+sc)^n\, f^{(n+1)}(c+s(x-c))\, ds =$$

$$= \frac{1}{n!}\int_0^1 (x-c)((x-sx)+(sc-c))^n\, f^{(n+1)}(c+s(x-c))\, ds =$$

$$= \frac{1}{n!}\int_0^1 (x-c)(x(1-s)-c(1-s))^n\, f^{(n+1)}(c+s(x-c))\, ds =$$

$$= \frac{1}{n!}\int_0^1 (x-c)(x-c)^n\,(1-s)^n\, f^{(n+1)}(c+s(x-c))\, ds =$$

$$= \frac{(x-c)^{n+1}}{n!}\int_0^1 (1-s)^n\, f^{(n+1)}(c+s(x-c))\, ds$$

Thus, we conclude that the integral form of the Taylor series expansion remainder can be expressed like below:

$$R_{n,c}(x) = \frac{(x-c)^{n+1}}{n!}\int_0^1 (1-s)^n\, f^{(n+1)}(c+s(x-c))\, ds$$

c)

The remainder can also be conveyed in the Lagrange form of the remainder.

Proof:

From Problem 1, we take the following theorem

$$f(\xi)\cdot\int_a^b g(x)\ dx = \int_a^b f(x)\, g(x)\, dx\ , \qquad for\ \xi \in [a,b]$$

It is clear that the left-hand side is the derivative of the function $F(\tau)$, which means that

$$\lim_{h \to 0} f(c) = F'(\tau)$$

For the theorem to hold true, the right-hand side should be equal to the function under the integral. To prove that the statement above is true we can use the Squeeze Law. By Squeeze Law: $c \in [\tau, \tau + h], \ \ so \ \ \tau \leq c \leq \tau + h$

$$\lim_{h \to 0} \tau = \tau$$

and

$$\lim_{h \to 0} \tau + h = \tau \, ,$$

Then this means that

$$\lim_{h \to 0} c = \tau$$

Since f is a continuous function at τ

$$\lim_{h \to 0} f(c) = f(\tau)$$

and this leads us to

$$F'(\tau) = f(\tau)$$

Consequently, the proof is concluded.