Orthonormal basis

$|e_j\rangle$ – unit vector.

$\langle e_i|e_j\rangle = \delta_{ij}$ ← we call this an orthonormal basis.

$$|a\rangle = \sum_{j=1}^{n} \alpha_j |e_j\rangle$$

$$\langle e_i|a\rangle = \sum_{j=1}^{n} \alpha_j \underbrace{\langle e_i|e_j\rangle}_{\delta_{ij}} = \alpha_i$$

*
$$\boxed{\alpha_j = \langle e_j|a\rangle}$$

$$\langle a|b\rangle = \sum_{i=1}^{n} \langle e_i|\alpha_i^* \sum_{j=1}^{n} \beta_j |e_j\rangle =$$

$$= \sum \alpha_i^* \beta_j \underbrace{\langle e_i|e_j\rangle}_{\delta_{ij}} = \sum_{j=1}^{n} \alpha_j^* \beta_j$$

*
$$\boxed{\begin{aligned} |a\rangle &= \sum_{j=1}^{n} \alpha_j |e_j\rangle \\ \langle a| &= \sum_{j=1}^{n} \langle e_j| \alpha_j^* \end{aligned}}$$

Building an orthonormal basis

$\{|a_j\rangle\}$

(1) $|e_1\rangle = \dfrac{|a_1\rangle}{|a_1|}$

(2) $|b_2\rangle = |a_2\rangle - |e_1\rangle\langle e_1|a_2\rangle$

$$\langle e_1|b_2\rangle = \langle e_1|a_2\rangle - \underbrace{\langle e_1|e_1\rangle}_{=1}\langle e_1|a_2\rangle$$

$$= 0$$

$$|e_2\rangle = \frac{|b_2\rangle}{|b_2|}$$