In the examples below, we shall illustrate some basic ideas involved in proof by induction.

EXAMPLE 1.2.1. We shall prove by induction that

$$1+2+3+\ldots+n=\frac{n(n+1)}{2} \tag{1}$$

for every $n \in \mathbb{N}$. To do so, let $p(n)$ denote the statement (1). Then clearly $p(1)$ is true. Suppose now that $p(n)$ is true, so that

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}.$$

Then

$$1+2+3+\ldots+n+(n+1)=\frac{n(n+1)}{2}+(n+1)=\frac{(n+1)(n+2)}{2},$$

so that $p(n+1)$ is true. It now follows from the Principle of induction (Weak form) that (1) holds for every $n \in \mathbb{N}$.

EXAMPLE 1.2.2. We shall prove by induction that

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6} \tag{2}$$

for every $n \in \mathbb{N}$. To do so, let $p(n)$ denote the statement (2). Then clearly $p(1)$ is true. Suppose now that $p(n)$ is true, so that

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

Then

$$\begin{aligned}1^2+2^2+3^2+\ldots+n^2+(n+1)^2&=\frac{n(n+1)(2n+1)}{6}+(n+1)^2=\frac{(n+1)(n(2n+1)+6(n+1))}{6}\\&=\frac{(n+1)(2n^2+7n+6)}{6}=\frac{(n+1)(n+2)(2n+3)}{6},\end{aligned}$$

so that $p(n+1)$ is true. It now follows from the Principle of induction (Weak form) that (2) holds for every $n \in \mathbb{N}$.

EXAMPLE 1.2.3. We shall prove by induction that $3^n > n^3$ for every $n > 3$. To do so, let $p(n)$ denote the statement

$$(n \le 3) \text{ or } (3^n > n^3).$$

Then clearly $p(1), p(2), p(3), p(4)$ are all true. Suppose now that $n > 3$ and $p(n)$ is true. Then $3^n > n^3$. It follows that (note that we are aiming for $(n+1)^3 = n^3+3n^2+3n+1$ all the way)

$$\begin{aligned}3^{n+1}&>3n^3=n^3+2n^3>n^3+6n^2=n^3+3n^2+3n^2>n^3+3n^2+6n\\&=n^3+3n^2+3n+3n>n^3+3n^2+3n+1=(n+1)^3,\end{aligned}$$

so that $p(n+1)$ is true. It now follows from the Principle of induction (Weak form) that $3^n > n^3$ holds for every $n > 3$.

PROOF. (a) Write $z = x + y\mathrm{i}$, where $x, y \in \mathbb{R}$. Then $z\overline{z} = (x + y\mathrm{i})(x - y\mathrm{i}) = x^2 + y^2$.

(b) Write $z = x + y\mathrm{i}$ and $w = u + v\mathrm{i}$, where $x, y, u, x \in \mathbb{R}$. Then $zw = (xu - yv) + (xv + yu)\mathrm{i}$, so that

$$|zw|^2 = (xu - yv)^2 + (xv + yu)^2 = (x^2 + y^2)(u^2 + v^2) = |z|^2|w|^2.$$

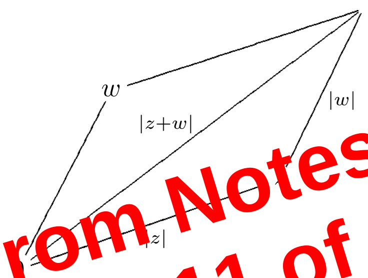
The result follows on taking square roots.

(c) Note that the result is trivial if $z + w = 0$. Suppose now that $z + w \neq 0$. Then

$$\begin{aligned}\frac{|z| + |w|}{|z + w|} &= \frac{|z|}{|z + w|} + \frac{|w|}{|z + w|} = \left|\frac{z}{z + w}\right| + \left|\frac{w}{z + w}\right| \\ &\geq \mathfrak{Re}\frac{z}{z + w} + \mathfrak{Re}\frac{w}{z + w} = \mathfrak{Re}\left(\frac{z}{z + w} + \frac{w}{z + w}\right) = \mathfrak{Re}1 = 1.\end{aligned}$$

The result follows immediately. ◯

REMARK. Proposition 1D(c) is known as the Triangle inequality. It can be understood easily from the diagram below:



The inequality follows on noting that the sum of the lengths of two sides of a triangle is at least the length of the third side.

We have shown earlier that the cartesian coordinates (x, y) are very useful for adding two complex numbers, whereas multiplication of complex numbers has a rather messy formula in cartesian coordinates. Let us use polar coordinates instead.

Suppose that

$$z = r(\cos\theta + \mathrm{i}\sin\theta) \qquad \text{and} \qquad w = s(\cos\phi + \mathrm{i}\sin\phi),$$

where $r, s, \theta, \phi \in \mathbb{R}$ and $r, s > 0$. Then

$$\begin{aligned}zw &= rs(\cos\theta + \mathrm{i}\sin\theta)(\cos\phi + \mathrm{i}\sin\phi) \\ &= rs((\cos\theta\cos\phi - \sin\theta\sin\phi) + \mathrm{i}(\cos\theta\sin\phi + \sin\theta\cos\phi)) \\ &= rs(\cos(\theta + \phi) + \mathrm{i}\sin(\theta + \phi)). \qquad (9)\end{aligned}$$

It follows that if we represent complex numbers in polar coordinates, then multiplication of complex numbers simply means essentially multiplying the moduli and adding the arguments. On the other hand, it is not difficult to show that

$$\frac{z}{w} = \frac{r}{s}(\cos(\theta - \phi) + \mathrm{i}\sin(\theta - \phi)). \qquad (10)$$

To study (16), note that $1+\sqrt{3}\mathrm{i} = 2(\cos(\pi/3) + \mathrm{i}\sin(\pi/3))$. It follows from Proposition 1F that the roots of (16) are given by

$$z = \sqrt[3]{2}\left(\cos\left(\frac{\pi}{9}+\frac{2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{9}+\frac{2k\pi}{3}\right)\right), \quad \text{where } k = 0,1,2;$$

in other words,

$$z_1 = \sqrt[3]{2}\left(\cos\frac{\pi}{9} + \mathrm{i}\sin\frac{\pi}{9}\right), \quad z_2 = \sqrt[3]{2}\left(\cos\frac{7\pi}{9} + \mathrm{i}\sin\frac{7\pi}{9}\right), \quad z_3 = \sqrt[3]{2}\left(\cos\frac{13\pi}{9} + \mathrm{i}\sin\frac{13\pi}{9}\right).$$

To study (17), note that $1-\sqrt{3}\mathrm{i} = 2(\cos(5\pi/3) + \mathrm{i}\sin(5\pi/3))$. It follows from Proposition 1F that the roots of (17) are given by

$$z = \sqrt[3]{2}\left(\cos\left(\frac{5\pi}{9}+\frac{2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{5\pi}{9}+\frac{2k\pi}{3}\right)\right), \quad \text{where } k = 0,1,2;$$

in other words,

$$z_4 = \sqrt[3]{2}\left(\cos\frac{5\pi}{9} + \mathrm{i}\sin\frac{5\pi}{9}\right), \quad z_5 = \sqrt[3]{2}\left(\cos\frac{11\pi}{9} + \mathrm{i}\sin\frac{11\pi}{9}\right), \quad z_6 = \sqrt[3]{2}\left(\cos\frac{17\pi}{9} + \mathrm{i}\sin\frac{17\pi}{9}\right).$$

1.8. Analytic Geometry

In classical analytic geometry, we express the equation of a locus as a relation between x and y. If we write $z = x + \mathrm{i}y$, then such an equation can be equally well described as a relation between z and $\overline{z}$. However, it is important to bear in mind that a complex equation is usually equivalent to two real equations, since each of the real part and the imaginary part of the complex equation gives rise to a real equation. It follows that to obtain a genuine locus, these two equations should be essentially the same. We also study some simple regions on the complex plane.

Here, we shall restrict our discussion to three examples. The reader is advised to draw some pictures.

EXAMPLE 1.8.1. The equation of a circle can be given by

$$|z-c| = r. \tag{18}$$

To see this, suppose that $z = x + \mathrm{i}y$ and $c = a + \mathrm{i}b$, where $x, y, a, b \in \mathbb{R}$. Then

$$|z-c|^2 = |(x+\mathrm{i}y) - (a+\mathrm{i}b)|^2 = |(x-a) + \mathrm{i}(y-b)|^2 = (x-a)^2 + (y-b)^2,$$

so that we have the equation $(x-a)^2 + (y-b)^2 = r^2$. Note that the equation (18) can also be written in the form

$$(z-c)(\overline{z}-\overline{c}) = r^2. \tag{19}$$

Note also that equation (19) is in invariant under conjugation; in other words, the conjugate of (19) is exactly the same as (19). Next, we consider the inequality $|z-c| < r$. A similar argument as above leads to the inequality $(x-a)^2 + (y-b)^2 < r^2$. This represents the region on the xy-plane inside the circle $(x-a)^2 + (y-b)^2 = r^2$. Similarly, the inequality $|z-c| > r$ represents the region on the xy-plane outside the circle $(x-a)^2 + (y-b)^2 = r^2$.