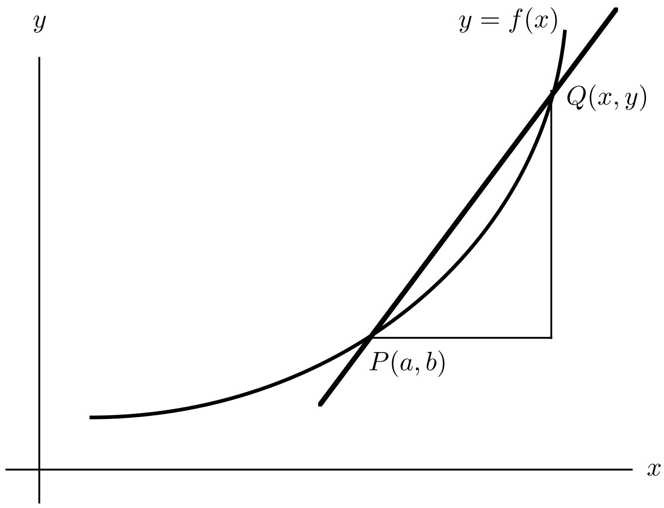
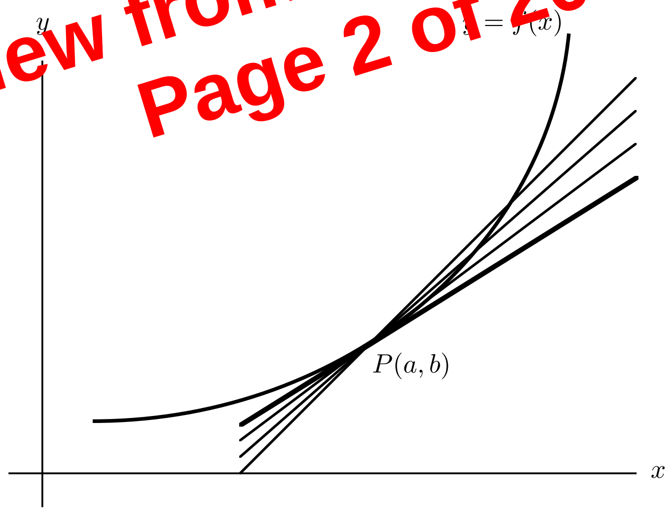
Consider the graph of a function $y = f(x)$. Suppose that $P(a, b)$ is a point on the curve $y = f(x)$. Consider now another point $Q(x, y)$ on the curve close to the point $P(a, b)$. We draw the line joining the points $P(a, b)$ and $Q(x, y)$, and obtain the picture below.



Clearly the slope of this line is equal to

$$\frac{y-b}{x-a} = \frac{f(x)-f(a)}{x-a}.$$

Now let us keep the point $P(a, b)$ fixed, and move the point $Q(x, y)$ along the curve towards the point P. Eventually the line PQ becomes the tangent to the curve $y = f(x)$ at the point $P(a, b)$, as shown in the picture below.



We are interested in the slope of this tangent line. Its value is called the derivative of the function $y = f(x)$ at the point $x = a$, and denoted by

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a} \qquad \text{or} \qquad f'(a).$$

In this case, we say that the function $y = f(x)$ is differentiable at the point $x = a$.

Remark. Sometimes, when we move the point $Q(x, y)$ along the curve $y = f(x)$ towards the point $P(a, b)$, the line PQ does not become the tangent to the curve $y = f(x)$ at the point $P(a, b)$. In this

EXAMPLE 3.6.2. Consider the function $y = f(x)$ described by the equation

$$2x^2y + \cos y = x^3.$$

Here, it is hard, if not impossible, to describe y explicitly in terms of x. However,

$$\frac{\mathrm{d}}{\mathrm{d}x}(2x^2y + \cos y) = \frac{\mathrm{d}}{\mathrm{d}x}(x^3).$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}x}(2x^2y + \cos y) = 4xy + 2x^2\frac{\mathrm{d}y}{\mathrm{d}x} - (\sin y)\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(x^3) = 3x^2,$$

we have

$$4xy + (2x^2 - \sin y)\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2.$$

EXAMPLE 3.6.3. We want to find the maximum value and minimum value of

$$z = x + 2y \tag{8}$$

subject to the constraint

$$x^2 + y^2 = 20. \tag{9}$$

Differentiating (8) and (9) with respect to x, we obtain respectively

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 1 + 2\frac{\mathrm{d}y}{\mathrm{d}x}$$

and

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{10}$$

When z is maximized or minimized, we must have $\mathrm{d}z/\mathrm{d}x = 0$, so that

$$1 + 2\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{11}$$

Combining (10) and (11) and eliminating $\mathrm{d}y/\mathrm{d}x$, we obtain

$$2x - y = 0. \tag{12}$$

Combining (9) and (12), we obtain $x = \pm 2$. Obviously, $z = 10$ when $x = 2$, while $z = -10$ when $x = -2$. It follows that the maximum value of z is 10, and that the minimum value of z is -10.

17. Consider the ellipse $(x+1)^2 + 2(y-1)^2 = 6$.
    a) Determine the slope of the tangent at the point $(1, 2)$.
    b) Determine the slope of the tangent at the point $(1, 0)$.

18. Use L'Hopital's rule to find each of the following:

    a) $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^3}$ b) $\displaystyle\lim_{x\to 0} \frac{\tan x - x}{x^3}$ c) $\displaystyle\lim_{x\to 1} \frac{x^3 - 3x + 2}{x^3 + x^2 - 5x + 3}$