$$I = \frac{4}{3}\int_0^a y^3 dy$$

$$I = \frac{4}{3}\left[\frac{y^4}{4}\right]_0^a = \frac{4}{3}\frac{a^4}{4} = \frac{a^4}{3}$$

Problem:-3

Change the order of integration $\int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dy\, dx$ and evaluate it.

Solution:

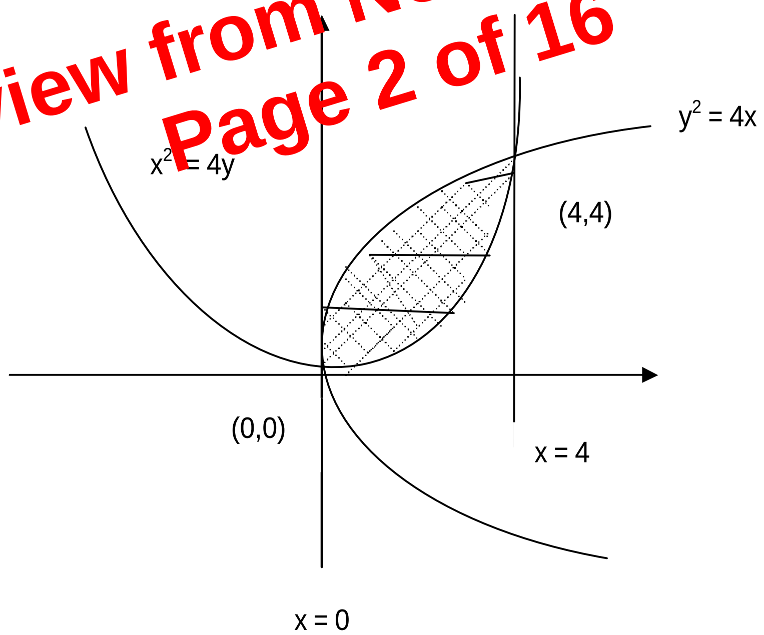
From the given data, first integral with respect to x and second with respect to y

$x = 0 \qquad y = \frac{x^2}{4} \qquad i.e.\ \ x^2 = 4y$

$x = 4 \qquad y = 2\sqrt{x} \qquad i.e.\ \ y^2 = 4x$

i.e the given order of integration is with respect to y first x second. [corresponds to vertical strip method]

$\therefore$ The region of integration is



By changing the order,

Let us consider horizontal strip method, we have the following limits

$x=y^2/4$, $x=2(y)^{1/2}$, $y=0$, $y=4$

The line x=1 divides the region R into R_1 and R_2. therefore we have $R=R_1+R_2$

$$I = \iint_{R_1} xydxdy + \iint_{R_2} xydxdy$$

R_1 (region in left side of the line x=1)

y=0, $y=(x)^{1/2}$

x=0,x=1

R_2(region in right side of the line x=1)

y=0, y=2-x

x=1,x=2

$$I = \int_0^1 \int_0^{\sqrt{x}} xy \, dy \, dx + \int_1^2 \int_0^{2-x} xy \, dy \, dx$$

$$= \int_0^1 x\left(\frac{y^2}{2}\right)_0^{\sqrt{x}} dx + \int_1^2 x\left(\frac{y^2}{2}\right)_0^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 x^2 \, dx + \frac{1}{2}\int_1^2 x(2-x)^2 \, dx$$

$$= \frac{1}{2}\int_0^1 x^2 \, dx + \frac{1}{2}\int_1^2 4x + x^3 - 4x^2 \, dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3}\right)_0^1 + \frac{1}{2}\left(2x^2 + \frac{x^4}{4} - \frac{4x^3}{3}\right)_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left(8+4-\frac{32}{3}\right)-\left(2+\frac{1}{4}-\frac{4}{3}\right)\right]$$

$$I = -\frac{23}{12}$$