Example 3: Factor $27x^3 + 125$

Solution

$$27x^3 + 125 = (3x)^3 + 5^3$$

Now use the Sum of Cubes Formula:

Ans $\boxed{(3x+5)(9x^2 - 15x + 25)}$

Example 4: Factor $8x^3 - 27y^3$

Solution

$$8x^3 - 27y^3 = (2x)^3 - (3y)^3$$

Now use the Difference of Cubes Formula:

Ans $\boxed{(2x-3y)(4x^2 + 6xy + 9y^2)}$

Chapter 1

Review of Functions

1.1 Functions

A. Definition of a Function

Every valid input, x, produces **exactly one** output, y; no more, no less

B. Explicit vs. Implicit Functions

1. **Explicit Functions:** function whose defining equation is solved for y.

2. **Implicit Functions:** function whose defining equation is **not** solved for y.

Example 2: $f(x) = |x|$

Solution

$$f(-x) = |-x| = |-1|\,|x| = |x|$$

$$f(x) = |x|$$

$$-f(x) = -|x|$$

Ans Since $f(-x) = f(x)$, f is **even**.

Example 3: $f(x) = (x-1)^2$

Solution

$$f(-x) = (-x-1)^2 = x^2 + 2x + 1$$

$$f(x) = (x-1)^2 = x^2 - 2x + 1$$

$$-f(x) = -(x-1)^2 = -(x^2 - 2x + 1) = -x^2 + 2x + 1$$

Ans Since $f(-x)$ is neither $f(x)$ nor $-f(x)$, f is **neither** even nor odd.

Example 2: Given $g(x) = x^3$, decompose g into even and odd parts.

Solution

Note: $\mathit{dom}\ g = (-\infty, \infty)$, so the domain is symmetric.

Now use the formulas:

$$\begin{aligned} g_{\text{even}}(x) &= \frac{1}{2}\left[g(x) + g(-x)\right] \\ &= \frac{1}{2}\left[x^3 + (-x)^3\right] \\ &= \frac{1}{2}\left[x^3 + (-1)^3 x^3\right] \\ &= \frac{1}{2}(x^3 - x^3) = 0. \end{aligned}$$

$$\begin{aligned} g_{\text{odd}}(x) &= \frac{1}{2}\left[g(x) - g(-x)\right] \\ &= \frac{1}{2}\left[x^3 - (-x)^3\right] \\ &= \frac{1}{2}\left[x^3 - (-1)^3 x^3\right] \\ &= \frac{1}{2}(x^3 + x^3) = x^3. \end{aligned}$$

Ans
$$\boxed{\begin{aligned} g_{\text{even}}(x) &= 0 \\ g_{\text{odd}}(x) &= x^3 \end{aligned}}$$

This was no surprise, really. g was already odd.

Note: This is another even/odd test. To test a function, do the decomposition . . .

If the even part is 0, the function is **odd**. If the odd part is 0, the function is **even**.

If neither are 0, the function is **neither** even nor odd.

Exercises

1. Determine if f is even, odd, or neither where

   a. $f(x) = 3x^2 + 2$

   b. $f(x) = 2x^2 - x + 1$

   c. $f(x) = x^3 - 2x$

   d. $f(x) = \sqrt{|x|}$

   e. $f(x) = x^{\frac{1}{3}}$

   f. $f(x) = \frac{1}{1+x^2}$

   g. $f(x) = \frac{x}{1+x^2}$

2. Given $f(x) = (x - 3)^2$, decompose f into even and odd parts.

3. Explain why a nonzero function can not have x-axis symmetry.

4. Explain why the domain of a function must be symmetric in order to be able to decompose it into even and odd parts.

Example 2: Are f and g inverses, where $f(x) = x^2$ and $g(x) = \sqrt{x}$?

Solution

Check the two conditions!

1. $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$

2. $(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = |x|$

Both conditions are not met, so . . .

Ans $\boxed{f \text{ and } g \text{ are NOT inverses}}$

Example 3: Determine if k is one-to-one where $k(x) = \frac{x^2+4}{x^2-3}$

Solution

1. Set $k(x) = k(a)$: $\frac{x^2+4}{x^2-3} = \frac{a^2+4}{a^2-3}$

2. Solve for x: LCD=$(x^2 - 3)(a^2 - 3)$, and $x \neq \sqrt{-3}, \sqrt{3}$

$$(x^2 - 3)(a^2 - 3)\left[\frac{x^2+4}{x^2-3}\right] = (x^2 - 3)(a^2 - 3)\left[\frac{a^2+4}{a^2-3}\right]$$

$$(a^2 - 3)(x^2 + 4) = (x^2 - 3)(a^2 + 4)$$

$$a^2x^2 + 4a^2 - 3x^2 - 12 = a^2x^2 + 4x^2 - 3a^2 - 12$$

$$4a^2 - 3x^2 = 4x^2 - 3a^2 \;\Rightarrow\; -7x^2 = -7a^2 \;\Rightarrow\; x^2 = a^2 \;\Rightarrow\; x = \pm a$$

Ans Since $x = \pm a$, not just $x = a$, k is not one-to-one

Exercises

Use the formal method to determine if f is one-to-one where

1. $f(x) = 2x - 7$.

2. $f(x) = x^2 + 3$.

3. $f(x) = \sqrt{5 - x}$.

4. $f(x) = \frac{2x-3}{x+1}$.

5. $f(x) = \log_3(2x + 1)$.

6. $f(x) = \frac{1}{1+x^2}$

7. Show that all linear functions with nonzero slope are one-to-one.

8. Show that if f is even, then f is **not** one-to-one.

Chapter 2

Rational Functions

2.1 The Reciprocal Function

Let $f(x) = \frac{1}{x}$. f is called the **reciprocal function**.

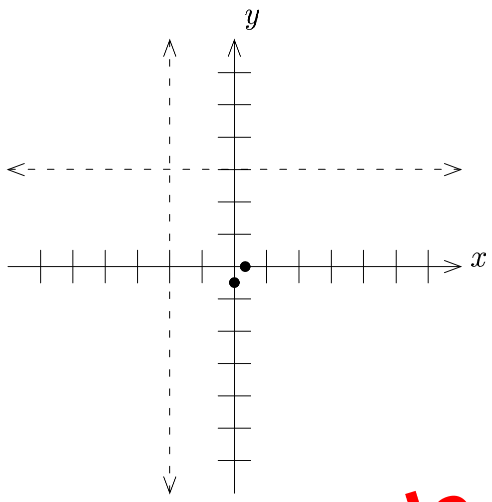
We can plot this by making a table of values. Since f is undefined at $x = 0$, we pick a lot of points near 0.

x	y
-10	$-\frac{1}{10}$
-3	$-\frac{1}{3}$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$-\frac{1}{3}$	-3
$-\frac{1}{10}$	-10
0	undefined

x	y
0	undefined
$\frac{1}{10}$	10
$\frac{1}{3}$	3
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$
10	$\frac{1}{10}$

y-intercept: set $x = 0$: $f(0) = \frac{3(0)-1}{0+2} = -\frac{1}{2}$

Now graph an initial rough sketch:



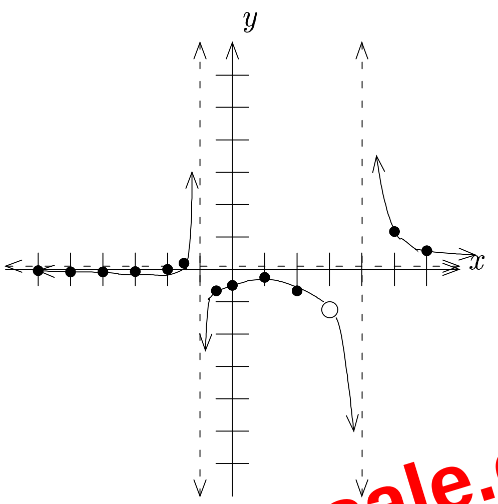
Now we need to plot enough points to see what's going on. We pick $x = -6$ and $x = 6$ to see the behavior near the horizontal asymptote, and pick $x = -3$ and $x = -1$ to see the behavior near the vertical asymptote. Then pick a few others to see what is going on:

x	y
-6	$\frac{19}{4}$
-4	$\frac{13}{2}$
-3	10
-1	-4
1	$\frac{2}{3}$
6	$\frac{17}{8}$

We plot these points on the grid we already made. Then we connect the points using the asymptote behavior.

We plot these points on the grid we already made. Then we connect the points using the asymptote behavior.

Ans



Example 3: Graph f, where $f(x) = \dfrac{x^3 + x^2 - 2}{x^2 + x + 1}$

Solution

1. **Asymptotes**:

   We first have to factor . . .

   Considering $x^2 + x + 1$, we can't factor it immediately, so we decide to use the quadratic formula. However $b^2 - 4ac = (1)^2 - 4(1)(1) = -3 < 0$, so the zeros are complex. Thus we have no vertical asymptotes or holes!

   Since degree top$=3$ and degree bottom$=2$, and since $3 > 2$, we have an oblique or curvilinear asymptote (oblique, in fact, as we see below).

Now use the quadratic formula on the remaining quadratic:

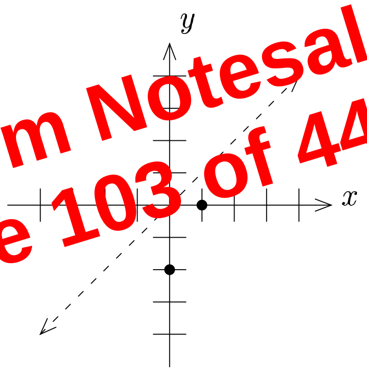
$$x = \frac{-2 \pm \sqrt{4-4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

Since these are complex, we only get one x-intercept from the $x - 1$ factor.

Thus we have one x-intercept, $x = 1$.

y-intercept: set $x = 0$: $\quad f(0) = \frac{0^3+0^2-2}{0^+0+1} = -2.$

Now graph an initial rough sketch:



Chapter 3

Elementary Trigonometry

3.1 Circles and Revolutions

A. Circles

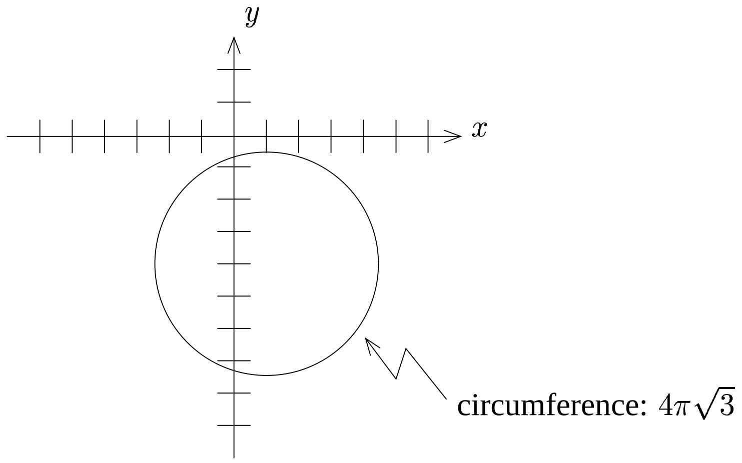
Standard Form: $(x - a)^2 + (y - b)^2 = r^2$

1. center: (a, b)

2. radius: r

3. circumference: $2\pi r$

Graph:



Example 2: Find the center, radius, circumference, x and y intercepts of the circle, where $x^2 + y^2 = 1$. Then sketch the circle.

Solution

center: $\boxed{(0,0)}$

radius: $\sqrt{1} = \boxed{1}$

circumference: $2\pi \cdot 1 = \boxed{2\pi}$

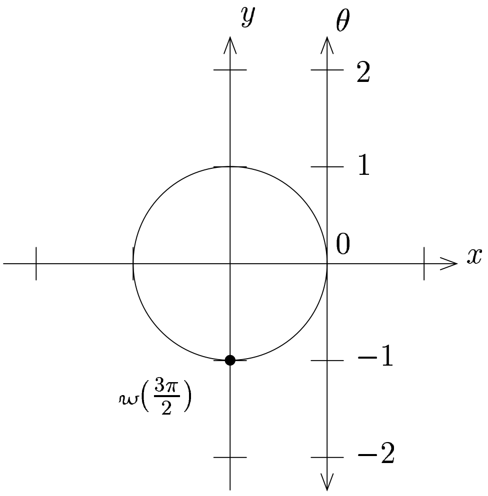
x**-intercepts:** set $y = 0$:

$$\begin{aligned} x^2 + 0^2 &= 1 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

$$\boxed{\text{Thus the } x\text{-intercepts are } \pm 1.}$$

Example 3: Evaluate $w(\frac{3\pi}{2})$

Solution



Ans $\boxed{w(\frac{3\pi}{2}) = (0, -1)}$

3.4 The Wrapping Function At Multiples of $\pi/4$

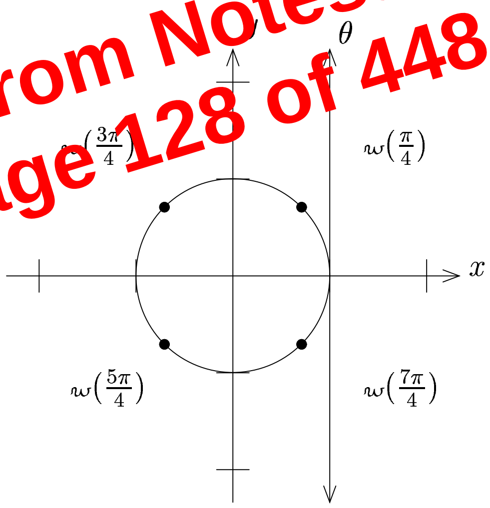
A. Introduction

Evaluating $w(\theta)$ for θ being a multiple of π or $\frac{\pi}{2}$ is direct. However, we need a rule for evaluating $w(\theta)$ when θ is a multiple of $\frac{\pi}{4}$.

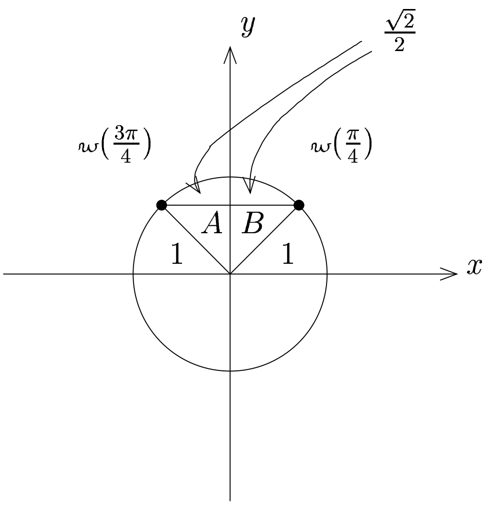
We will derive the $\frac{\pi}{4}$ rule in six easy steps.

B. Derivation of the $\frac{\pi}{4}$ Rule

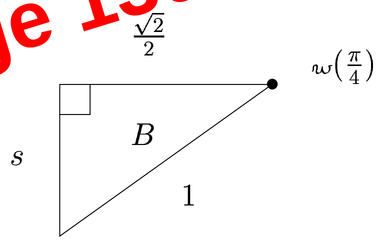
Step 1: Note that if θ is a multiple of $\frac{\pi}{4}$ (lowest terms), then $w(\theta)$ is in one of 4 spots.



Step 4: By symmetry, the y axis bisects the triangle into two with top edge length $\frac{\sqrt{2}}{2}$



Step 5: We examine triangle B.

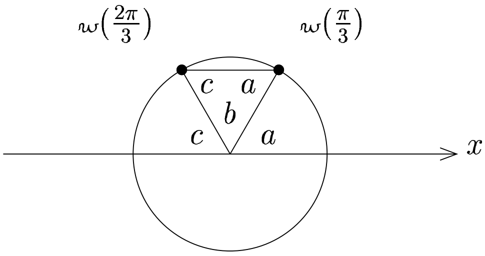


We can use the Pythagorean Theorem again to find s:

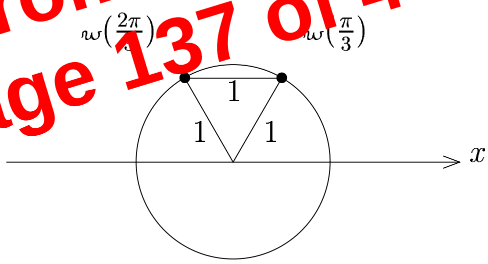
$$s^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = 1^2 \Rightarrow s^2 + \frac{2}{4} = 1 \Rightarrow s^2 = \frac{2}{4} \Rightarrow s = \frac{\sqrt{2}}{2}$$

Step 3: Alternate Interior Angles Of Two Parallel Lines
Cut By A Transversal Are Congruent

By the above geometric fact, the other internal angles of the triangle are c and a respectively, as in the diagram.



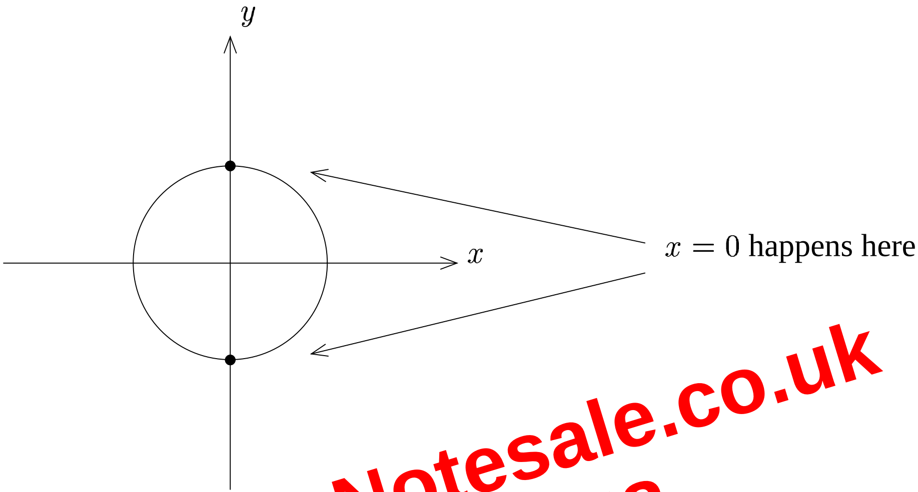
Step 4: Since $\angle a \cong \angle b \cong \angle c$, the triangle is equiangular. Thus the triangle is equilateral, and all sides have length 1.



B. Tangent

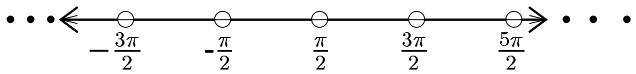
1. **Domain:**

Given $w(\theta) = (x, y)$, we have $\tan\theta = \frac{y}{x}$. Now $\frac{y}{x}$ is undefined when $x = 0$. When does this happen?



Thus $\tan\theta$ is undefined for $\theta = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

What is this in interval notation? To see it, let's plot the allowed values on a number line:



Thus $dom(\tan)$: $\ldots \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \ldots$

Note: Each interval has an endpoint being an "odd multiple of $\frac{\pi}{2}$".

Since $2k + 1$ is the formula that generates odd numbers (for k an integer), we recognize that

$dom(\tan)$: union of all intervals of the form $\left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right)$, where $k \in \mathbb{Z}$
[k is an integer]

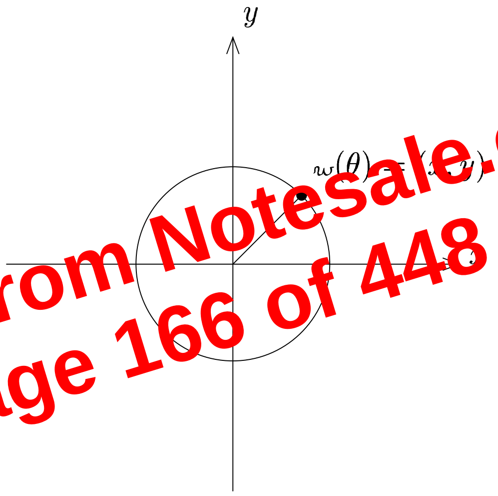
C. Quotient Identities

Using the definitions again, we get

1. $\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$

2. $\boxed{\cot\theta = \frac{\cos\theta}{\sin\theta}}$

D. The Pythagorean Identity



Note: $x^2 + y^2 = 1$ (because we have a unit circle)

Since we have that $\cos\theta = x$ and $\sin\theta = y$, the equation becomes

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

Shorthand: $(\cos\theta)^2 = \cos^2\theta$

Warning: $\cos\theta^2$ **does not** mean $(\cos\theta)^2$; $\cos\theta^2$ means $\cos(\theta^2)$

Exercises

1. Know $\cos\theta = \frac{1}{3}$. Find $\sec\theta$.

2. Know $\sin\theta = -\frac{1}{5}$. Find $\csc(-\theta)$.

3. Know $\cos\theta = \frac{2}{7}$. Find $\sec(-\theta)$.

4. If $\sin\theta = \frac{1}{4}$, what are the possible values of $\cos\theta$?

5. If $\cos\theta = \frac{2}{3}$, what are the possible values of $\sin\theta$?

6. If $\tan\theta = 3$, what are the possible values of $\sec\theta$?

7. If $\csc\theta = -10$, what are the possible values of $\cot\theta$?

8. If $\sin\theta = 1$, what are the possible values of $\cos\theta$?

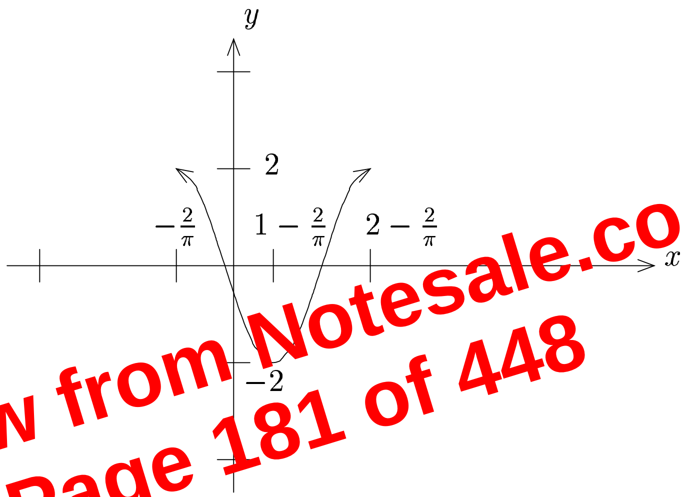
Example 2: Graph f, where $f(x) = 3 + 2\cos(\pi x + 2)$

Solution

1. $\pi x + 2 = 0 \;\Rightarrow\; x = -\frac{2}{\pi}$

2. $\pi x + 2 = 2\pi \;\Rightarrow\; \pi x = 2\pi - 2 \;\Rightarrow\; x = 2 - \frac{2}{\pi}$

3. Note below that the y-intercept before the vertical shift, being $2\cos(2)$, is negative.



4. There is no reflection

5. Shift up 3 to get final answer

Exercises

1. Graph f, where

   a. $f(x) = 3\cos 4x$

   b. $f(x) = 2\sin(x - \frac{\pi}{4})$

   c. $f(x) = 5\cos(2x - \frac{\pi}{3})$

   d. $f(x) = -\frac{1}{2}\cos(\pi x + \frac{\pi}{2})$

   e. $f(x) = 2 + 3\sin(x + \frac{\pi}{3})$

   f. $f(x) = 4\cos(3x - 2) - 5$

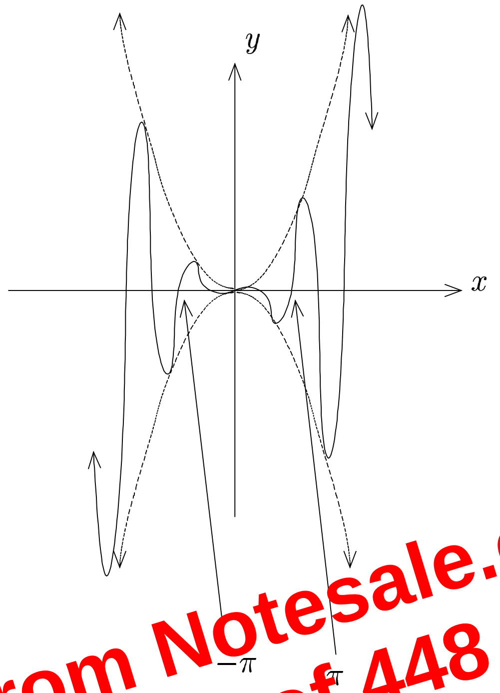
   g. $f(x) = 1 - \frac{1}{3}\cos(\frac{\pi}{4}x - \frac{\pi}{3})$

   h. $f(x) =$ [illegible] $+$ [illegible]$/2 \, \sin(\frac{3\pi}{2}x + \frac{\pi}{4})$

2. Let $f(x) = 2 + 3\cos(\pi x + 1)$. Find $\text{rng}\, f$.

Now draw in the damping curves $y = x^2$ and $y = -x^2$, then modify:

Ans



Example 2: Graph f, where $f(x) = e^x \cos 3x$

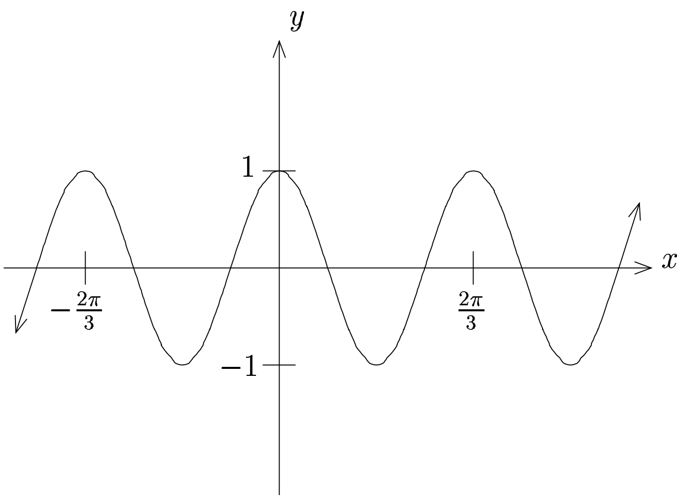
Solution

First graph $y = \cos 3x$:

a. $3x = 0 \Rightarrow x = 0$

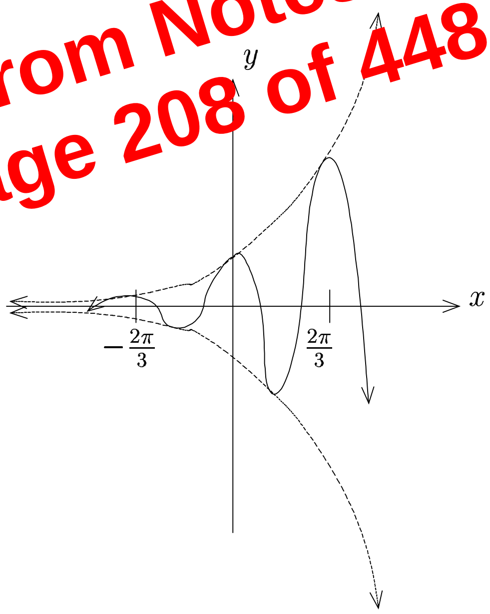
b. $3x = 2\pi \Rightarrow x = \frac{2\pi}{3}$

c.



Now draw in the damping curves $y = e^x$ and $y = -e^x$, then modify

Ans

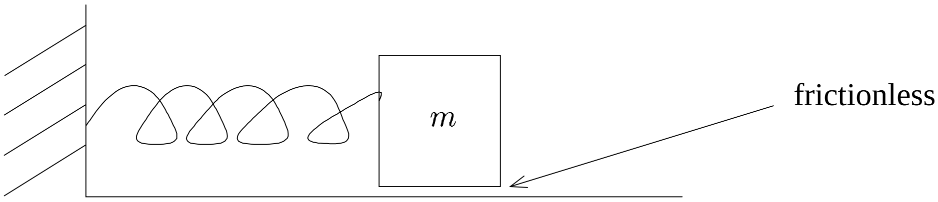


4.8 Simple Harmonic Motion and Frequency

A. Simple Harmonic Motion

An object that oscillates in time uniformly is said to undergo **simple harmonic motion**.

Example: Spring-Mass System



Here $d = a \sin(\omega t)$ or $d = a \cos(\omega t)$, where

d: displacement from equilibrium position

a: maximum displacement

ω: angular frequency

B. Frequency

1. **Period,** T: $T = \frac{2\pi}{\omega}$ time to undergo one complete cycle

   Units: units of time, typically seconds (s)

2. **Frequency,** ν: $\nu = \frac{1}{T}$ "oscillation speed" (how many cycles per time)

   Units: inverse units of time, typically s^{-1}, also called Hertz (Hz)

Example 2: Find a model for simple harmonic motion satisfying the conditions:

- Period: 6s
- Maximum Displacement: 2m
- Displacement at $t = 0$: 2m

Solution

Since the object starts at maximum displacement, we use the cosine model:

$$d = a \cos(\omega t)$$

Now $a = 2$, and $T = 6 = \frac{2\pi}{\omega}$, so $\omega = \frac{2\pi}{6} = \frac{\pi}{3}$.

Ans $\boxed{d = 2 \cos(\frac{\pi}{3} t)}$

Example 3: Factor $\sec^2 x + 5\tan x + 5$

Solution

Can't factor directly, so convert to same trigonometric function!

Use Pythagorean II: $1 + \tan^2 x = \sec^2 x$

Thus,

$\sec^2 x + 5\tan x + 5$

$= (1 + \tan^2 x) + 5\tan x + 5$

$= \tan^2 x + 5\tan x + 6$

$= (\tan x + 2)(\tan x + 3)$

Ans $\boxed{(\tan x + 2)(\tan x + 3)}$

Example 3: Verify the identity: $\dfrac{1}{1+\sin\theta}+\dfrac{1}{1-\sin\theta}=2\sec^2\theta$

Solution

Start with the left side:

$$\frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}$$

$$=\frac{1-\sin\theta}{(1+\sin\theta)(1-\sin\theta)}+\frac{1+\sin\theta}{(1+\sin\theta)(1-\sin\theta)} \qquad \text{(LCD)}$$

$$=\frac{2}{(1+\sin\theta)(1-\sin\theta)} \qquad \text{(adding)}$$

$$=\frac{2}{1-\sin^2\theta} \qquad \text{(multiply out bottom)}$$

$$=\frac{2}{(\cos^2\theta+\sin^2\theta)-\sin^2\theta} \qquad \text{(use Pythagorean I)}$$

$$=\frac{2}{\cos^2\theta}$$

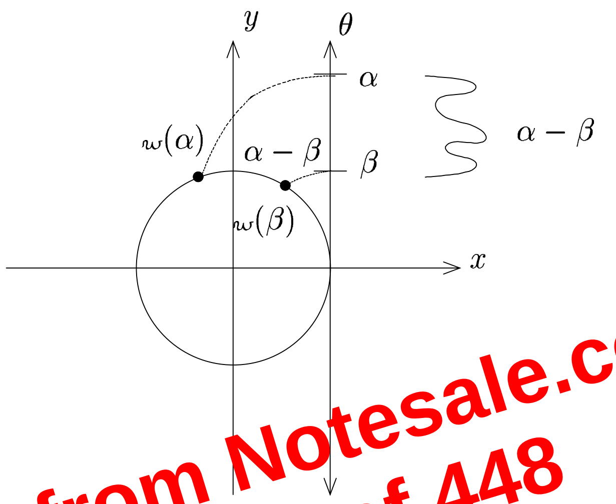
$$=2\sec^2\theta \qquad \text{(use reciprocal identity)}$$

Thus we reached the right side, so we are done.

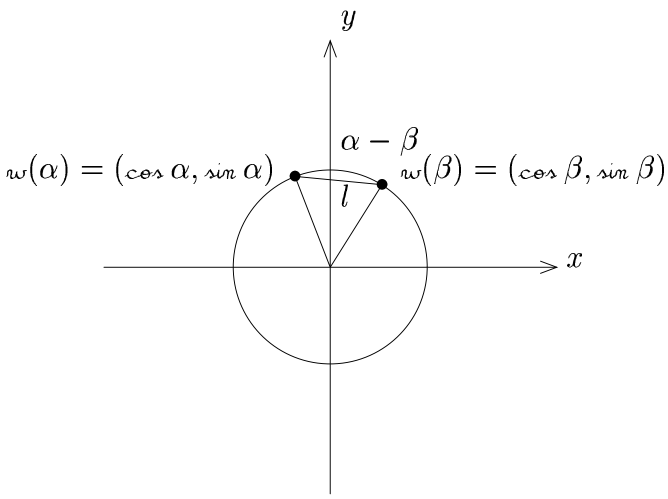
5.3 Sum and Difference Formulas I

A. Derivation of $\cos(\alpha - \beta)$

Step 1: For values α and β on the number line, identify $w(\alpha)$, $w(\beta)$, and $\alpha - \beta$



Step 2: Connect points to form triangle, and calculate length l (distance between $w(\alpha)$ and $w(\beta)$



Distance Formula: $l = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$

F. Formula for $\tan(\alpha \pm \beta)$

Writing $\tan(\alpha \pm \beta)$ as $\dfrac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)}$, and then expanding and simplifying (Exercise), we get

$$\boxed{\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}}$$

Comments:

1. The above formula will only work when $\tan \alpha$ and $\tan \beta$ are defined!

2. If they are not defined, then you need to simplify the expression the long way, using

$$\boxed{\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)}}$$

Example 2: Find $\tan\left(\frac{\pi}{12}\right)$

Solution

Write $\frac{\pi}{12}$ as $\frac{\pi}{3} - \frac{\pi}{4}$!

Then

$$\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{4}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{4}\right)}$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3} \cdot 1}$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \cdot \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= \frac{3 - 2\sqrt{3} + 1}{3 - 1}$$

$$= \frac{4 - 2\sqrt{3}}{2}$$

$$= \frac{2(2 - \sqrt{3})}{2}$$

$$= 2 - \sqrt{3}$$

Ans $\boxed{2 - \sqrt{3}}$

Example 2: Express $\cos 7\theta - \cos 4\theta$ as a product

Solution

Use $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$:

Thus

$$\cos 7\theta - \cos 4\theta = -2\sin\left(\frac{7\theta + 4\theta}{2}\right)\sin\left(\frac{7\theta - 4\theta}{2}\right)$$

$$= -2\sin\left(\frac{11\,\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)$$

Ans $\boxed{-2\sin\left(\frac{11\,\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)}$

Example 2: Verify the identity: $\dfrac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta} = \tan\left(\dfrac{\alpha+\beta}{2}\right)\cot\left(\dfrac{\alpha-\beta}{2}\right)$

Solution

$$\frac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta}$$

$$= \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)} \qquad \text{(sum to product formulas)}$$

$$= \frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)} \cdot \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$= \tan\left(\frac{\alpha+\beta}{2}\right)\cot\left(\frac{\alpha-\beta}{2}\right)$$

Exercises

Verify the following trigonometric identities:

1. $\frac{\cos\theta - \cos 3\theta}{\sin\theta + \sin 3\theta} = \tan\theta$

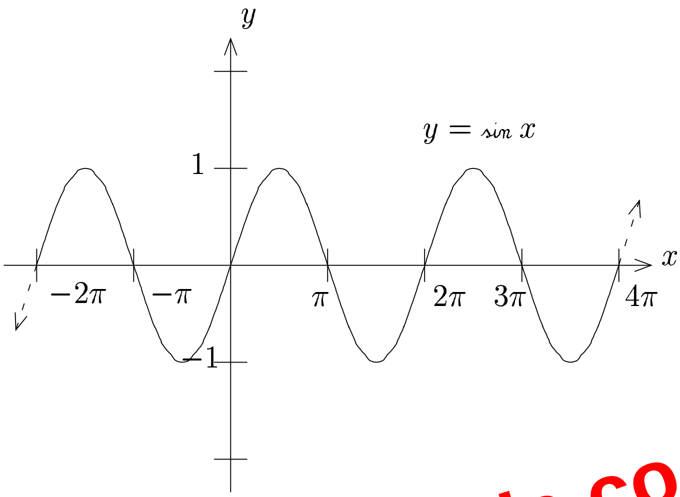
2. $\frac{\cos 4\theta - \cos 2\theta}{2\sin 3\theta} = -\sin\theta$

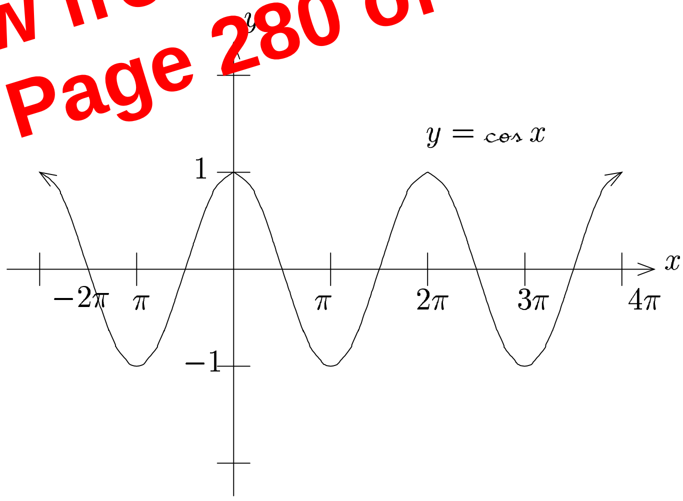
3. $\frac{\cos\alpha + \cos\beta}{\cos\alpha - \cos\beta} = -\cot\left(\frac{\alpha+\beta}{2}\right)\cot\left(\frac{\alpha-\beta}{2}\right)$

4. $\frac{\cos\theta - \cos 5\theta}{\sin\theta + \sin 5\theta} = \tan 2\theta$

B. The Six Trigonometric Functions

To motivate what comes next, let us first review the graphs of the six trigonometric functions.



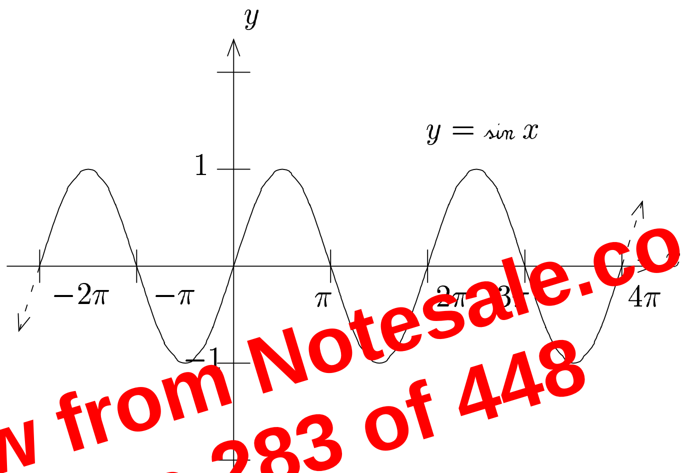


C. Motivation

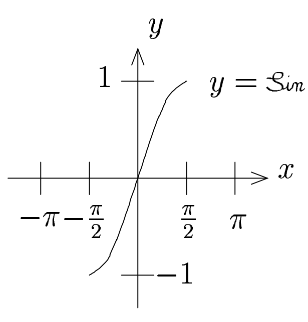
All six trigonometric functions fail the horizontal line test, so are **not** one-to-one/invertible.

We therefore define the capital trigonometric functions.

D. Capital Sine Construction



To make this function one-to-one, **without changing the range**, one choice that can be made is to throw away everything except the part of the graph between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. This is not the only choice, but it is the most obvious choice.



The residual function is a capital function. We call it $\mathcal{S}in$.

Thus $\mathcal{S}in\ x = sin\ x;\ \ x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

2. To get $\sin\theta$ from $\cos\theta$, we use $\cos^2\theta + \sin^2\theta = 1$:

$$\left(-\frac{1}{4}\right)^2 + \sin^2\theta = 1$$

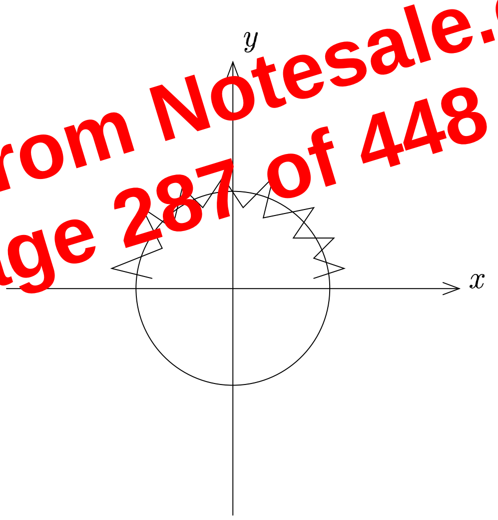
$$\frac{1}{16} + \sin^2\theta = 1$$

$$\sin^2\theta = \frac{15}{16}$$

$$\sin\theta = \pm\frac{\sqrt{15}}{4}$$

3. Use the restricted domain to try to remove the sign ambiguity:

Since $\theta \in [0, \pi]$, we are in the region marked:



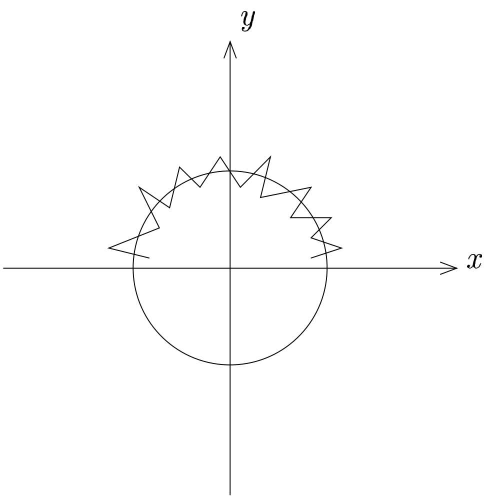
Here $\sin\theta \geq 0$, so

Ans $\boxed{\sin\theta = \frac{\sqrt{15}}{4}}$

---

3. Use the restricted domain to try to remove the sign ambiguity:

Since $\theta \in [0, \pi]$, we are in the region marked:



Here $\sin\theta \geq 0$, so $\sin\theta = \frac{2\sqrt{6}}{5}$

However, our original problem was to find $\sin 2\theta$.

Thus,

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
&= (2)\left(\frac{2\sqrt{6}}{5}\right)\left(-\frac{1}{5}\right) \\
&= -\left(\frac{4\sqrt{6}}{5}\right)\left(\frac{1}{5}\right) \\
&= -\frac{4\sqrt{6}}{25}
\end{aligned}$$

Ans $\boxed{\sin 2\theta = -\frac{4\sqrt{6}}{25}}$

Then

$$\cot^2\theta + 1 = \csc^2\theta$$

$$\cot^2\theta + 1 = \left(-\frac{5}{3}\right)^2$$

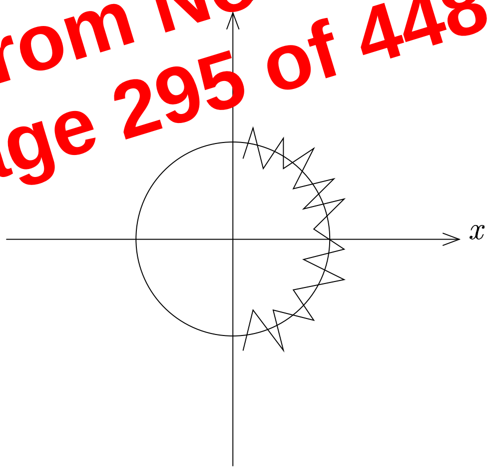
$$\cot^2\theta + 1 = \frac{25}{9}$$

$$\cot^2\theta = \frac{16}{9}$$

$$\cot\theta = \pm\frac{4}{3}$$

3. Use the restricted domain to try to remove the sign ambiguity:

Since $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we are in the region marked:



We see that $\cot\theta \geq 0$ in quadrant I but $\cot\theta \leq 0$ in quadrant IV.

Thus we have no **initial** help!

However, since we were originally given $\sin\theta = -\frac{3}{5}$.

Thus $\sin\theta < 0$.

D. Comments

1. **Warnings:**

a. "$^{-1}$" means inverse function when attached to functions, **not** reciprocal

$\mathrm{Sin}^{-1}x$ means inverse sine of x

$\dfrac{1}{\mathrm{Sin}\,x}$ takes the values of cosecant

Note: These are different.

b. Sin^{-1} and sin are **not** inverses! sin does not have an inverse!
The functions that are inverses are Sin^{-1} and Sin. Be careful of this in problems.

c. Some authors are lazy and write sin^{-1}, when they really mean Sin^{-1}.
To avoid confusion, write Sin^{-1} if that is what is intended.

2. In some older books, Sin^{-1}, Cos^{-1}, Tan^{-1}, Cot^{-1}, Sec^{-1}, Csc^{-1} are sometimes written as Arcsin, Arccos, Arctan, Arccot, Arcsec, and Arccsc. In that context, inverse sine, Sin^{-1}, is pronounced "arc-sine" when it is written as Arcsin.

E. Evaluation

We can evaluate inverse trigonometric functions if the output is a multiple of π, $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, or $\frac{\pi}{6}$. To do so, we look for appropriate combinations/ratios of $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, $\frac{\sqrt{2}}{2}$, etc.

Remember the range of the inverse trigonometric function!

6.5 Inverse Trigonometric Problems

A. Method of Solution

1. Define the inverse trigonometric function output to be θ.

2. Rewrite the θ definition with no inverse trigonometric function by applying the appropriate capital trigonometric function to each side.

3. Recast the problem as a capital trigonometric function problem, and solve it.

B. Examples

Example 1: Find $\text{sin}(\text{Cos}^{-1}(\frac{2}{3}))$

Solution

1. Let $\theta = \text{Cos}^{-1}(\frac{2}{3})$.

2. Then $\text{Cos}\,\theta = \frac{2}{3}$.

3. Thus we have the capital trigonometric problem:

You know $\text{Cos}\,\theta = \frac{2}{3}$. Find $\text{sin}\,\theta$.

a. $\text{Cos}\,\theta = \frac{2}{3}$

$\text{cos}\,\theta = \frac{2}{3};\ \theta \in [0, \pi]$

Example 3: Find $\cos(2\tan^{-1}(\frac{3}{4}))$

Solution

1. Let $\theta = \tan^{-1}(\frac{3}{4})$.

2. Then $\tan\theta = \frac{3}{4}$.

3. Thus we have the capital trigonometric problem:

   You know $\tan\theta = \frac{3}{4}$. Find $\cos 2\theta$.

   a. $\tan\theta = \frac{3}{4}$

   $\tan\theta = \frac{3}{4};\ \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

   b. Now $\cos(2\theta) = 2\cos^2\theta - 1$, so we need $\cos\theta$.

   However $1 + \tan^2\theta = \sec^2\theta$, so $1 + (\frac{3}{4})^2 = \sec^2\theta$.

   Thus $\sec^2\theta = 1 + \frac{9}{16} = \frac{25}{16}$.

   Then $\cos^2\theta = \frac{16}{25}$.

   In fact, we have no need for $\cos\theta$!

$\cos(\frac{\pi}{2} - \theta) = x;\ \frac{\pi}{2} - \theta \in [0, \pi]$

$\text{Cos}(\frac{\pi}{2} - \theta) = x$

$\frac{\pi}{2} - \theta = \text{Cos}^{-1} x$

Then solving for θ, we have $\theta = \frac{\pi}{2} - \text{Cos}^{-1} x$.

Hence we verified that $\text{Sin}^{-1} x = \frac{\pi}{2} - \text{Cos}^{-1} x$, so

$$\boxed{\text{Sin}^{-1} x + \text{Cos}^{-1} x = \frac{\pi}{2}}$$

However, the sum of functions method is useful since it provides a way to tackle identities that you can't figure out the other way.

Example 2: Verify the identity: $\text{Tan}^{-1}\left(\frac{1}{4}\right) + \text{Tan}^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{4}$

Solution:

Since $\tan$ is more natural here . . .

1. Simplify $\tan\left(\text{Tan}^{-1}\left(\frac{1}{4}\right) + \text{Tan}^{-1}\left(\frac{3}{5}\right)\right)$:

   Let $\theta_1 = \text{Tan}^{-1}\left(\frac{1}{4}\right)$ and $\theta_2 = \text{Tan}^{-1}\left(\frac{3}{5}\right)$.

   Then $\text{Tan}\,\theta_1 = \frac{1}{4}$ and $\text{Tan}\,\theta_2 = \frac{3}{5}$.

   Hence we have the following capital trigonometric problem to solve:

   Know $\text{Tan}\,\theta_1 = \frac{1}{4}$ and $\text{Tan}\,\theta_2 = \frac{3}{5}$. Find $\tan(\theta_1 + \theta_2)$.

Now $\operatorname{Tan}\theta_1 = \frac{1}{4} \Rightarrow \tan\theta_1 = \frac{1}{4};\ \theta_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$

and $\operatorname{Tan}\theta_2 = \frac{3}{5} \Rightarrow \tan\theta_2 = \frac{3}{5};\ \theta_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Also

$$\begin{aligned}
\tan(\theta_1 + \theta_2) &= \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} \\
&= \frac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{1}{4}\cdot\frac{3}{5}} \\
&= \frac{\frac{5}{20} + \frac{12}{20}}{1 - \frac{3}{20}} \\
&= \frac{\frac{17}{20}}{\frac{17}{20}} \\
&= 1
\end{aligned}$$

Hence we have the identity:

$$\tan\left(\operatorname{Tan}^{-1}\left(\tfrac{1}{4}\right) + \operatorname{Tan}^{-1}\left(\tfrac{3}{5}\right)\right) = 1$$

2. Now $\theta_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\theta_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, so $\theta_1 + \theta_2 \in (-\pi, \pi)$.

Thus we have that $\operatorname{Tan}^{-1}\left(\frac{1}{4}\right) + \operatorname{Tan}^{-1}\left(\frac{3}{5}\right) \in (-\pi, \pi)$.

Since $\operatorname{Tan}^{-1}\left(\frac{1}{4}\right) + \operatorname{Tan}^{-1}\left(\frac{3}{5}\right) \in (-\pi, \pi)$ and $\tan\left(\operatorname{Tan}^{-1}\left(\frac{1}{4}\right) + \operatorname{Tan}^{-1}\left(\frac{3}{5}\right)\right) = 1$, and the only values of $\theta \in (-\pi, \pi)$ whose tangent is 1 is $-\frac{3\pi}{4}$ and $\frac{\pi}{4}$, we have that

$$\operatorname{Tan}^{-1}\left(\tfrac{1}{4}\right) + \operatorname{Tan}^{-1}\left(\tfrac{3}{5}\right) = -\tfrac{3\pi}{4} \quad \text{or} \quad \operatorname{Tan}^{-1}\left(\tfrac{1}{4}\right) + \operatorname{Tan}^{-1}\left(\tfrac{3}{5}\right) = \tfrac{\pi}{4}.$$

It only remains to determine which of the two identities is correct.

3. **Reflection Identities**

a. $\boxed{\mathrm{Sin}^{-1}(-x) = -\mathrm{Sin}^{-1}x}$

b. $\boxed{\mathrm{Cos}^{-1}(-x) = \pi - \mathrm{Cos}^{-1}x}$

c. $\boxed{\mathrm{Tan}^{-1}(-x) = -\mathrm{Tan}^{-1}x}$

d. $\boxed{\mathrm{Cot}^{-1}(-x) = \pi - \mathrm{Cot}^{-1}x}$

e. $\boxed{\mathrm{Sec}^{-1}(-x) = \pi - \mathrm{Sec}^{-1}x}$

f. $\boxed{\mathrm{Csc}^{-1}(-x) = -\mathrm{Csc}^{-1}x}$

B. Calculator Use

Since many calculators don't have all six inverse trigonometric functions on them, we can use the above identities to do computations in calculators.

In particular,

1. $\mathrm{Cot}^{-1}x = \frac{\pi}{2} - \mathrm{Tan}^{-1}x$

2. $\mathrm{Sec}^{-1}x = \mathrm{Cos}^{-1}\left(\frac{1}{x}\right)$

3. $\mathrm{Csc}^{-1}x = \mathrm{Sin}^{-1}\left(\frac{1}{x}\right)$

reduces the evaluation of inverse trigonometric functions to that of inverse sine, inverse cosine, and inverse tangent.

In fact, using the identity, $\mathrm{Tan}^{-1}x = \mathrm{Sin}^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$, we can reduce the need to that of an **inverse sine button** only!

Then

1. $\mathcal{C}os^{-1}x = \frac{\pi}{2} - \mathcal{S}in^{-1}x$

2. $\mathcal{T}an^{-1}x = \mathcal{S}in^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

3. $\mathcal{C}ot^{-1}x = \frac{\pi}{2} - \mathcal{S}in^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

4. $\mathcal{S}ec^{-1}x = \frac{\pi}{2} - \mathcal{S}in^{-1}\left(\frac{1}{x}\right)$

5. $\mathcal{C}sc^{-1}x = \mathcal{S}in^{-1}\left(\frac{1}{x}\right)$

C. Strategy

1. Use algebra to isolate a trigonometric function on one side of the equation.

2. Find all solutions in $[0, 2\pi)$ through help from looking at the unit circle, and the definitions of the trigonometric functions.

3. The answer is obtained by taking each solution and adding "$2\pi k$" to get all solutions.

Note: In situations where more than one type of trigonometric function occurs in an equations, we try to either

a. separate the functions via factoring

or

b. get rid of one of the trigonometric functions via trigonometric identities.

D. Examples

Example 1: Solve $4\cos^2 x - 3 = 0$ for x

Solution

$$4\cos^2 x - 3 = 0$$

$$4\cos^2 x = 3$$

$$\cos^2 x = \frac{3}{4}$$

$$\cos x = \pm\frac{\sqrt{3}}{2}$$

$$x_{\text{coord}} = \pm\frac{\sqrt{3}}{2}$$

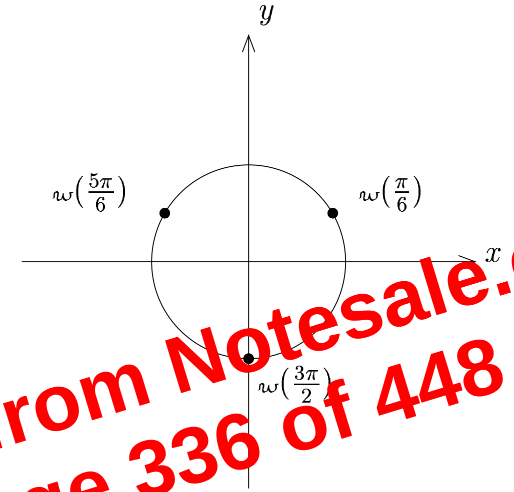
Thus $(\sin x + 1)(2\sin x - 1) = 0$.

By the Zero Product Principle:

$$\sin x + 1 = 0 \quad \text{or} \quad 2\sin x - 1 = 0$$

$$\sin x = -1 \quad \text{or} \quad \sin x = \tfrac{1}{2}$$

$$y_{\text{coord}} = -1 \quad \text{or} \quad y_{\text{coord}} = \tfrac{1}{2}$$



Ans $x = \begin{cases} \frac{\pi}{6} + 2\pi k; & k \in \mathbb{Z} \\ \frac{5\pi}{6} + 2\pi k; & k \in \mathbb{Z} \\ \frac{3\pi}{2} + 2\pi k; & k \in \mathbb{Z} \end{cases}$

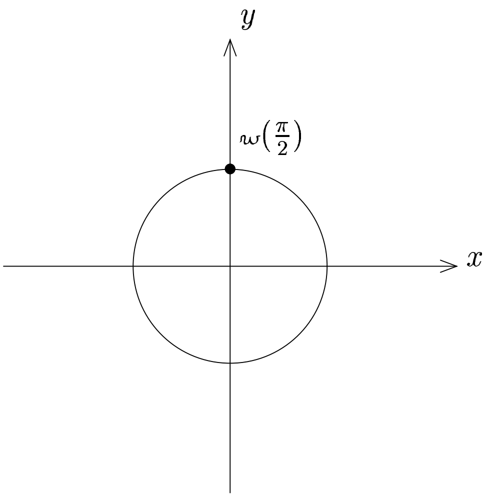
---

14. $\cos^2 x + \sin x = 2$

15. $\tan^3 x - \tan^2 x + 3\tan x - 3 = 0$

16. $\sin x = 1 - \cos x$

17. $\csc x + \cot x = 1$



Ans $\boxed{x = \frac{\pi}{2} + 2\pi k;\ k \in \mathbb{Z}}$

Example 2: Solve $2\cos x + \sin 2x = 0$ for x

Solution

$$2\cos x + \sin 2x = 0$$

$$2\cos x + 2\sin x\cos x = 0 \qquad \text{(double angle formula)}$$

$$2\cos x(1 + \sin x) = 0$$

By the Zero-Product Principle:

$$2\cos x = 0 \quad \text{or} \quad 1 + \sin x = 0$$

$$\cos x = 0 \quad \text{or} \quad \sin x = -1$$

Both of these solutions together, lie on the unit circle in the following locations:

Example 2: Graph f, where $f(x) = \sin x + \sqrt{3}\cos x$

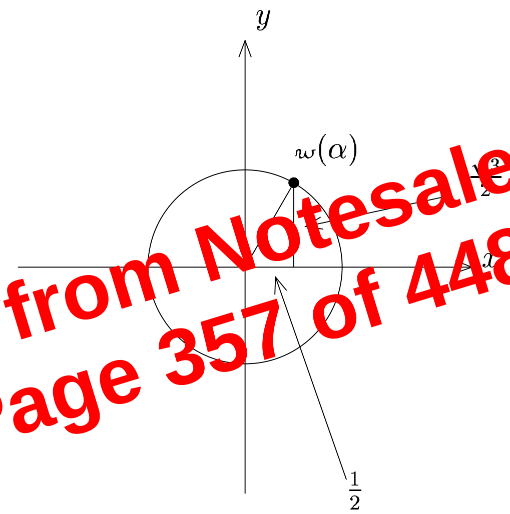
Solution

Compress the harmonic combination . . .

$$\sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

$$f(x) = 2\sin(x + \alpha), \text{ where } \alpha = \tan^{-1}(\sqrt{3})$$

In fact, we can simplify $\tan^{-1}(\sqrt{3})$:



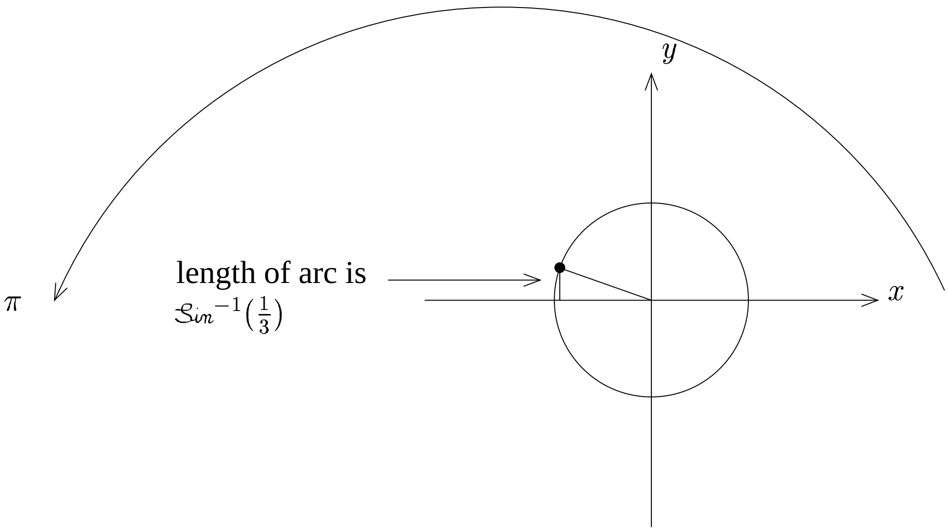
Thus $\alpha = \frac{\pi}{3}$.

Hence, we graph f, where $f(x) = 2\sin(x + \frac{\pi}{3})$

1. $x + \frac{\pi}{3} = 0 \;\Rightarrow\; x = -\frac{\pi}{3}$

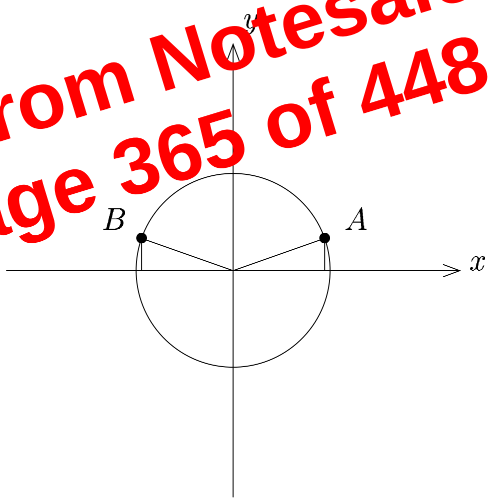
2. $x + \frac{\pi}{3} = 2\pi \;\Rightarrow\; x = \frac{5\pi}{3}$

Note that the y-intercept is $2\sin\frac{\pi}{3} = 2(\frac{\sqrt{3}}{2}) = \sqrt{3}$.



Thus B: $\pi - \mathcal{S}in^{-1}(\frac{1}{3}) + 2\pi k;\ k \in \mathbb{Z}$.

This solves the problem. However, let us reconsider the original picture:



Note: If we consider the two triangles, we know that the legs of the two triangle are congruent, since both have length $\frac{1}{3}$ and the hypotenuse of the two triangles are congruent, since both have length 1 (unit circle). Thus, by the HL Postulate, the two triangles are congruent. Thus the inner **angles** of the triangles are the same.

This suggests that learning information about angles would make this problem easier.

Goal: Connect arc length to angles.

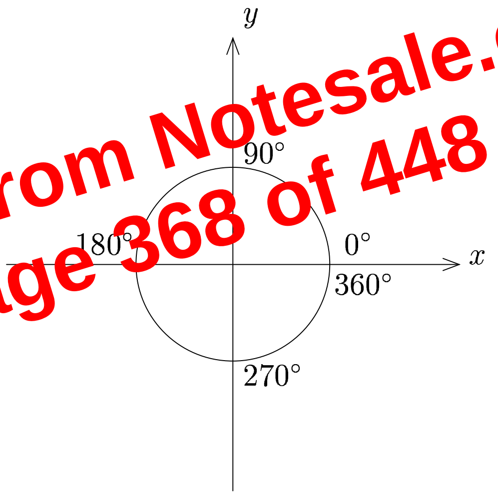
$$\frac{s}{C_2} = \frac{\theta}{C_1}$$

Thus $s = \theta(\frac{C_2}{C_1}) = \theta(\frac{2\pi r}{2\pi})$.

Hence, we have the **arc length formula**: $\boxed{s = r\theta}$

E. Degree Measure of Angles

The **degree measure** of an angle is defined by dividing up the angle of one complete revolution into 360°.



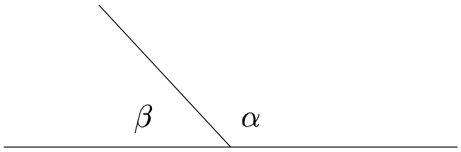
F. Conversion

We know 2π radians $= 360°$.

Thus $\frac{2\pi}{360°} = 1 \;\Rightarrow\; \frac{\pi}{180°} = 1$.

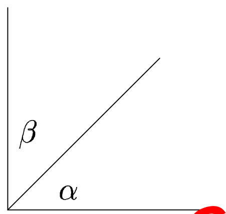
This gives us the following conversion rules:

2. **Supplementary Angles:** Angles that differ by π



$$\boxed{\beta = \pi - \alpha}$$

3. **Complementary Angles:** Angles that differ by $\frac{\pi}{2}$



$$\boxed{\beta = \frac{\pi}{2} - \alpha}$$

I. Examples Involving Arc Length

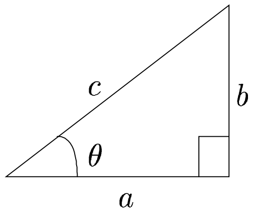
The arc length formula $s = r\theta$ assumes that angles are measured in radians. If an angle is given in degrees, we need to convert to radians first before using the arc length formula.

7.2 Right Triangle Trigonometry

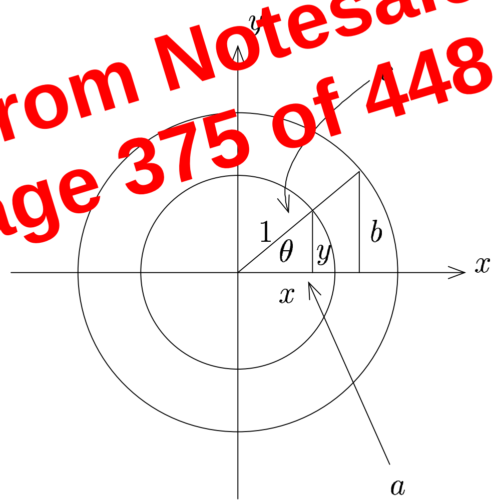
We now begin applications of trigonometry to geometry using angle ideas.

A. Development

Suppose we have a right triangle:



Draw circles of radius 1 and radius c :



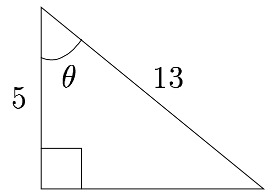
Note: By similar triangles, $\frac{a}{x} = \frac{c}{1}$ and $\frac{b}{y} = \frac{c}{1}$

Thus $x = \frac{a}{c}$ and $y = \frac{b}{c}$.

Hence $\cos\theta = \frac{a}{c}$ and $\sin\theta = \frac{b}{c}$.

Then we have that $\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$ and $\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$

Example 2: Given the right triangle:



Find $\sin\theta$, $\cos\theta$, $\tan\theta$

Solution

We first get the third side via the Pythagorean Theorem:

$$5^2 + s^2 = 13^2 \;\Rightarrow\; s^2 = 13^2 - 5^2 = 169 - 25 = 144 \;\Rightarrow\; s = 12$$

Now use the right triangle definitions:

$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} = \boxed{\frac{12}{13}}$$

$$\cos\theta = \frac{\text{adj.}}{\text{hyp.}} = \boxed{\frac{5}{13}}$$

$$\tan\theta = \frac{\text{opp.}}{\text{adj.}} = \boxed{\frac{12}{5}}$$

Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we may write

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

3. **Angle Formula**

If two lines are not perpendicular, and neither is vertical, then the smallest angle θ between the two lines is given by:

$$\boxed{\theta = \tan^{-1}\left(\left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|\right)}$$

4. **Example:** Find the smallest angle between the two lines given by $2x - y = 4$ and $3x + y = 3$.

Solution

Slope of $2x - y = 4$: $m_1 = 2$

Slope of $3x + y = 3$: $m_2 = -3$

Then

$$\theta = \tan^{-1}\left(\left|\frac{-3 - 2}{1 + (2)(-3)}\right|\right)$$

$$= \tan^{-1}\left(\left|\frac{-5}{-5}\right|\right)$$

$$= \tan^{-1}(1)$$

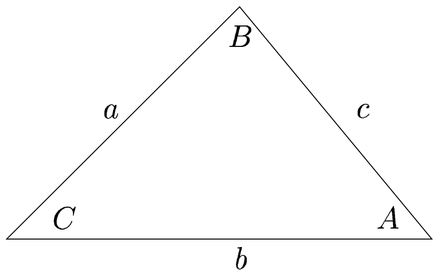
$$= \frac{\pi}{4}$$

Ans $\boxed{\frac{\pi}{4}}$ (or 45°)

7.4 Oblique Triangle Formulas and Derivations

We now consider triangles that are **not** right triangles. These are called **oblique triangles**.

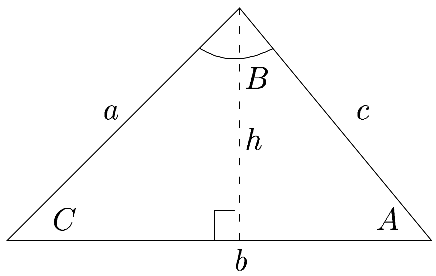
A. Law of Sines



1. **Law:** $\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$

2. **Derivation:**

Here we assume that we have an **acute triangle**, i.e. all angles in the triangle are acute. If the triangle is **obtuse** (i.e. an angle whose measure is greater than 90° exists in the triangle), then the derivation is similar.

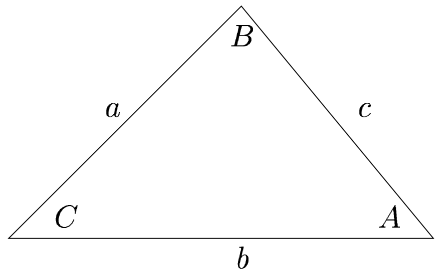


$\sin C = \frac{h}{a} \Rightarrow h = a \sin C$

$\sin A = \frac{h}{c} \Rightarrow h = c \sin A$

Thus $a \sin C = c \sin A$, so $\dfrac{\sin C}{c} = \dfrac{\sin A}{a}$.

C. Mollweide's Formulas



1. **Formulas:**

$$\boxed{\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)}}$$

$$\boxed{\frac{a-b}{c} = \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{C}{2}\right)}}$$

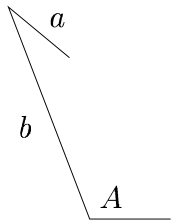
2. **Derivation:**

We'll derive the first version. The other can be derived similarly.

Since $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, we have that $\frac{\sin A}{\sin C} = \frac{a}{c}$ and $\frac{\sin B}{\sin C} = \frac{b}{c}$

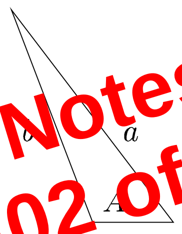
c. conditions for $\angle A$ obtuse:

I.



$a \leq b \Rightarrow$ no triangle

II.



$a > b \Rightarrow$ one triangle

7.6 Solving Oblique Triangles

A. Strategy

1. Given the side/angle data, draw a rough sketch of the triangle(s).

2. If appropriate, use Law of Sines. If not sufficient, use Law of Cosines.

3. Check your answers in one of Mollweide's Formulas (it doesn't matter which one). Some solutions may be fake, and this will tell you.

B. Tips

1. If possible, try to find the largest angle first. This is the angle opposite the longest side. This will tell you automatically that the other two angles are acute, and can help to eliminate fake solutions.

2. Remember that all three angles of a triangle add to 180°.

Now at II, the solution (as in the beginning of section 7.1), is $\pi - \text{Sin}\, C$, i.e. approx. $180° - 35.88° = 144.12°$.

Thus we have two cases, and two possible triangles (so far).

Case I: $C \approx 35.88°$

Then find B: $B \approx 180° - 23° - 35.88° = 121.12°$

Then find b:

Law of Sines: $\frac{\sin 121.12°}{b} = \frac{\sin 23°}{10}$

Thus, $b(\sin 23°) \approx 10 \sin 121.12°$

Then $b = \frac{10 \sin 121.12°}{\sin 23°} \approx 21.9.$

Case II: $C \approx 144.12°$

Then find B: $B \approx 180° - 23° - 144.12° = 12.88°$

Then find b:

Law of Sines: $\frac{\sin 12.88°}{b} = \frac{\sin 23°}{10}$

Thus, $b(\sin 23°) \approx 10 \sin 12.88°$

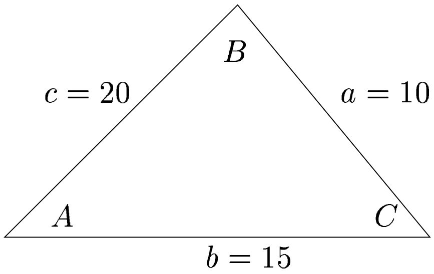
Then $b = \frac{10 \sin 12.88°}{\sin 23°} \approx 5.7.$

Now we need to check the answers using one of Mollweide's Formulas.

Example 3: Solve the triangle: $a = 10$, $b = 15$, $c = 20$

Solution

Draw a Picture:



Law of Sines won't work (yet).

Law of Cosines: Find C (largest angle)

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$20^2 = 10^2 + 15^2 - 2(10)(15) \cos C$$

$$2(10)(15) \cos C = 10^2 + 15^2 - 20^2$$

$$\cos C = \frac{10^2 + 15^2 - 20^2}{2(10)(15)} = -.25$$

Since $C \in [0, \pi]$, we have $\mathrm{Cos}\, C = -.25$, so $C = \mathrm{Cos}^{-1}(-.25) \approx 104.48°$

Note: Since we found the largest angle, we know that the other two angles are acute!

Find A:

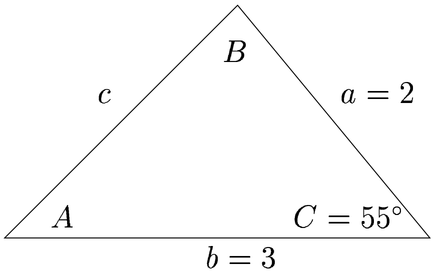
Now we can use the Law of Sines: $\frac{\sin 104.48°}{20} = \frac{\sin A}{10}$

Thus $\sin A = \frac{10 \sin 104.48°}{20} \approx .484$.

Example 4: Solve the triangle: $a = 2$, $b = 3$, $C = 55°$

Solution

Draw a Picture:



Law of Sines won't work (yet).

Find c:

Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$c^2 = 2^2 + 3^2 - 2(2)(3) \cos 55°$$

$$c^2 = 4 + 9 - 12 \cos 55°$$

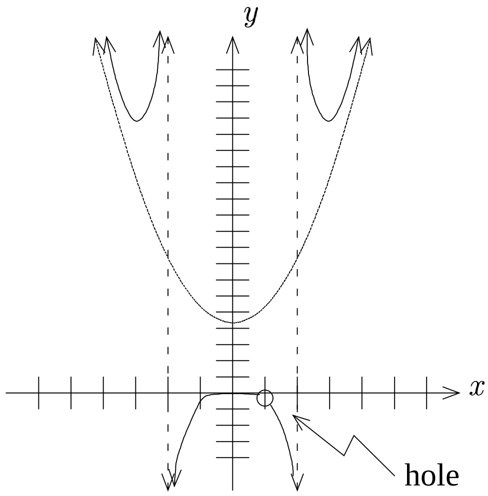
$$c = \sqrt{13 - 12 \cos 55°} \approx 2.473$$

Find B:

Law of Sines: $\frac{\sin B}{3} = \frac{\sin 55°}{2.473}$

Then $\sin B = \frac{3 \sin 55°}{2.473} \approx .994$.

4.



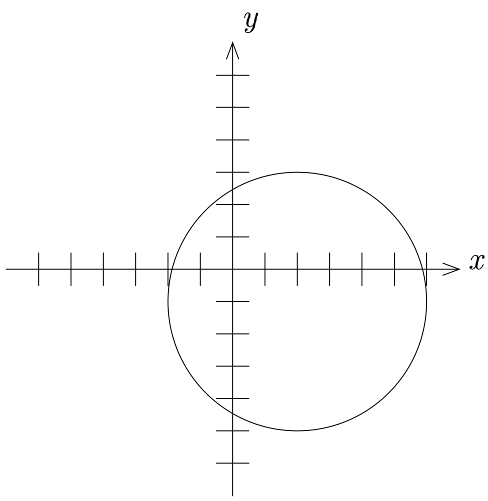
3.1

1a. $x^2 + y^2 = 4$

1c. $(x-3)^2 + (y+1)^2 = 7$

2b. center: $(2, -1)$; radius: 4; circumference: 8π

x-intercepts: $2 \pm \sqrt{15}$; y-intercepts: $-1 \pm 2\sqrt{3}$



4. $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$

6. $(0, -1)$

7. $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$

9. $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$

12. $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$

3.6

1a. $-\frac{\sqrt{2}}{2}$

1c. $\frac{2\sqrt{3}}{3}$

1e. -1

1g. $-\frac{1}{2}$

1i. $-\frac{\sqrt{2}}{2}$

1k. 2

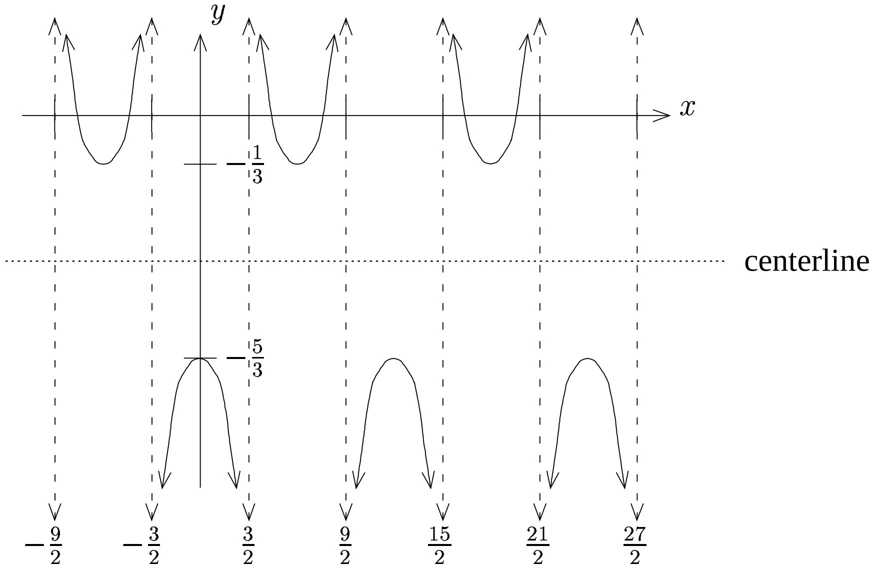
1m. undefined

1o. undefined

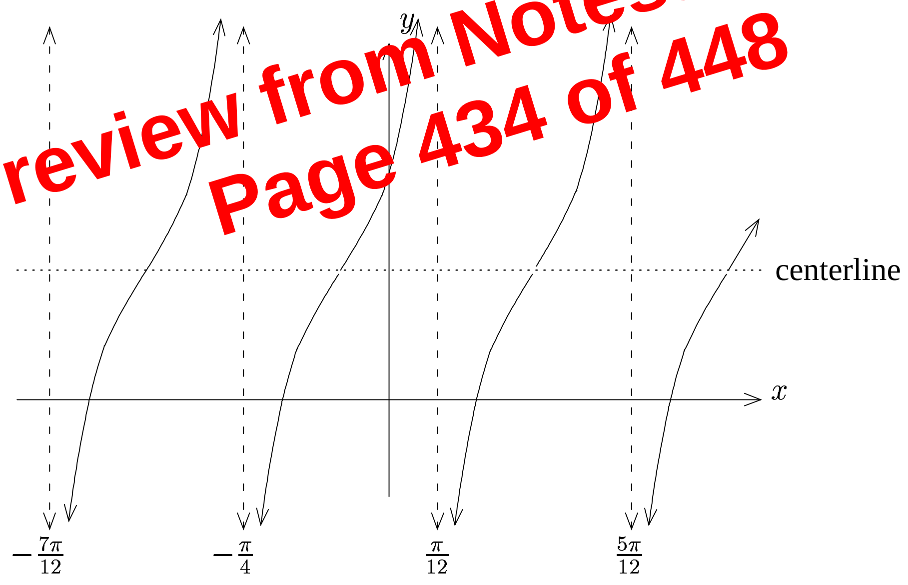
3.9

1. $-\frac{1}{4}$

2d.


y
x
$-\frac{1}{3}$
centerline
$-\frac{5}{3}$
$-\frac{9}{2}$
$-\frac{3}{2}$
$\frac{3}{2}$
$\frac{9}{2}$
$\frac{15}{2}$
$\frac{21}{2}$
$\frac{27}{2}$

2g.


y
centerline
x
$-\frac{7\pi}{12}$
$-\frac{\pi}{4}$
$\frac{\pi}{12}$
$\frac{5\pi}{12}$

4. $\frac{1-\sqrt{2}}{4}$

5. $\frac{\cos(2\theta)-\cos(8\theta)}{2}$

7. $\frac{\sin(6\theta)-\sin(8\theta)}{2}$

5.10

2. $-2\sin(3\theta)\sin(2\theta)$

4. $2\cos(3\theta)\cos\theta$

6.2

1. $\frac{\sqrt{7}}{4}$

2. $\frac{2\sqrt{2}}{3}$

4. $\frac{\sqrt{11}}{6}$

6.3

1. $-\frac{4\sqrt{21}}{125}$

3. $-\frac{1}{8}$

5. $-\sqrt{15}$

8. $\frac{4}{5}$

9. $2\sqrt{6}-5$

11. $\left(\frac{4\sqrt{17}}{17}, -\frac{\sqrt{17}}{17}\right)$

12. $\frac{5\sqrt{7}-2\sqrt{5}}{30}$

6.4

1. $\frac{\pi}{6}$

2. $-\frac{\pi}{6}$

4. $\frac{5\pi}{6}$

5. $\frac{\pi}{4}$

8. $\frac{5\pi}{11}$

11. $\frac{\pi}{3}$

6.5

1. $\frac{2\sqrt{6}}{5}$

4. $\frac{4\sqrt{7}}{7}$

5. $-\frac{7}{25}$

7. $2x^2 - 1$

8. 0

9. $\sqrt{15} - 4$

11. $\frac{24}{7}$

13. $\frac{8\sqrt{10}+9}{35}$

14. $\frac{4\sqrt{17}+2\sqrt{34}}{51}$

3. $x = \begin{cases} \frac{5\pi}{4} + 2\pi k;\ k \in \mathbb{Z} \\ \frac{7\pi}{4} + 2\pi k;\ k \in \mathbb{Z} \end{cases}$

5. $\emptyset$

6. $x = \begin{cases} \frac{\pi}{3} + 2\pi k;\ k \in \mathbb{Z} \\ \pi + 2\pi k;\ k \in \mathbb{Z} \\ \frac{5\pi}{3} + 2\pi k;\ k \in \mathbb{Z} \end{cases}$

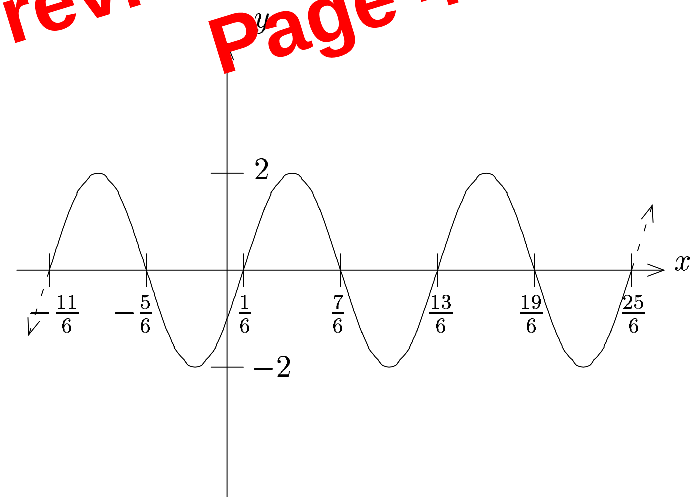
8. $x = \begin{cases} \frac{\pi}{15} + \frac{2\pi k}{15};\ k \in \mathbb{Z} \\ \frac{\pi}{3} + \frac{2\pi k}{3};\ k \in \mathbb{Z} \end{cases}$

6.10

1a. $5\sin(4\theta + \alpha)$, where $\alpha = \tan^{-1}(\frac{4}{3})$

1c. $2\sqrt{5}\sin(5\theta + \alpha)$, where $\alpha = \tan^{-1}(-2)$

2b.



3b. $x = \begin{cases} \frac{\pi}{15} + \frac{1}{5}\tan^{-1}(3) + \frac{2\pi k}{5};\ k \in \mathbb{Z} \\ \frac{2\pi}{15} + \frac{1}{5}\tan^{-1}(3) + \frac{2\pi k}{5};\ k \in \mathbb{Z} \end{cases}$