Each of the points 1,2,3… stands for the **end** of the period 1,2,3… that is, in the above diagram 3 should be read as end of period 3. The beginning of period 3 will be at point 2, which is the end of period 2.

The point 0 is the present or beginning of period 1. Similarly, the point n is the end of period n. The interval between i and i+1 should be read as “during the period i+1”.

For example, in the above diagram the interval between one and two should be read as during period 2.

3. Note that in number 2 we did not specify any particular “time unit” for the word period. Depending upon the problem at hand the period could be one year, one month, ten minutes, one million years. The general tendency is that when somebody mentions the word “period”, the students usually assume the period as one year long. Never assume that unless it is the rate of interest.

4. The rate of interest is always quoted in annual terms whether it is specifically mentioned or not.

   For example, if you call a mortgage broker to ask him/her the current rate of interest on a 30 year mortgage. He/she might say 4% without specifying 4% per year or per month or whatever. The very statement “rate of interest 4%” means that it is the annual rate. If it is not an annual rate, it will say so specified specifically. For example, if you borrow $100 from Mr. “machine-gun” Hernandez, the local crime boss, he might inform you “hey man, the raaate is 9 peercent per weeek, comprehende”. Then it is, of course, 9 percent per week.

5. In the computational formula (will come later), the rate of interest always enters in fractions.

   For example,
   10 % will enter as 0.10
   0.5 % will enter as 0.005
   If you are confused as what fraction to use, just divide the annual percentage rate by 100.

   For example,
   0.2% will be equal to 0.2/100 is 0.002 etc.

6. However, unlike the above convention (number 5), in case of using calculator the rate of interest is entered as percent.

   For example,

next n_2 years, and Pk_3n_3 for the next n_3 years, etc. This implies that the future value under changing simple interest rates is

$$FV = P(1 + n_1k_1 + n_2k_2 + n_3k_3 + ...)$$

Example 3.1: Suppose an account earns 15 percent simple interest per year. What would the future value be of a deposit of:

(a) \$15 left for 10 years?
(b) \$0.15 left for 5 years?
(c) \$1 left for 2,000 years?

Answers

$(a) \quad P = 15, k = .15, n = 10$

$$FV = 15(1 + .15 \times 10) = \$37.5$$

$(b)\ P = .15, k = .15, n = 5$

$$FV = .15(1 + .15 \times 5) = \$0.2625$$

$(c)\ P = 1, k = .15, n = 2,000$

$$FV = 1(1 + .15 \times 2,000) = \$301$$

Example 3.2: Suppose you deposit \$1,000 in an account pays simple interest. What will be the future value of the account if:

(a) the annual simple interest rate is 7% for the first 5 years, 10% for the next 10 years, and 12% for the last 5 years?
(b) 5% for the first 10 years, 10% for the next 10 years, 15% for the last 10 years?

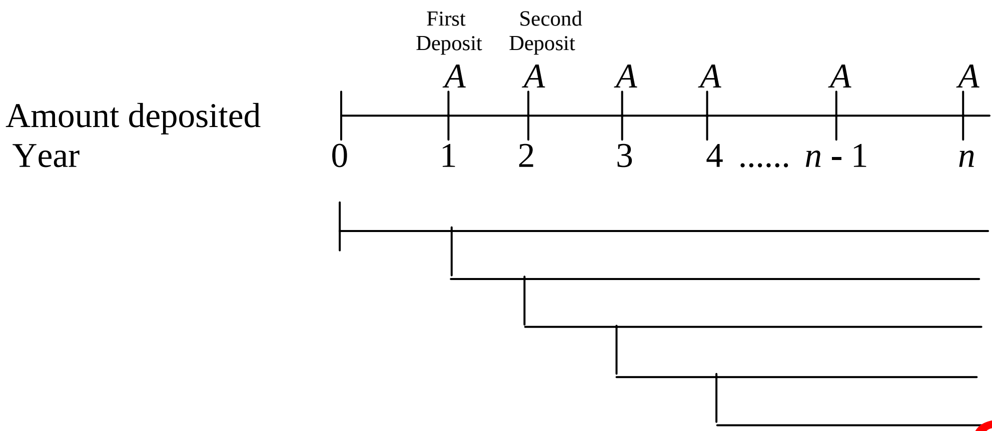
Answers:

(a) $P = \$1,000,\ k_1 = 7\%,\ n_1 = 5$
$k_2 = 10\%,\ n_2 = 10$
$k_3 = 12\%,\ n_3 = 5$
$FV = \$1,000(1 + .07 \times 5 + .10 \times 10 + .12 \times 5)$
$= \$1,000(2.95) = \$2,950.$

(b) $P = \$1,000,\ k_1 = 5\%,\ n_1 = 10$
$k_2 = 10\%,\ n_2 = 10$
$k_3 = 15\%,\ n_3 = 10$

Future Value of an Annuity

We are, of course, interested in the future value of a series of level deposits at the end of n years if the account earns an annual simple interest rate of k. The mathematical formula can be obtained with the help of the following diagram.



Applying formula (3.1) to each of the deposits, we have

$$FV \text{ of first } A = A[1+(n-1)k]$$

$$FV \text{ of second } A = A[1+(n-2)k]$$

$$FV \text{ of third } A = A[1+(n-3)k]$$

$$\vdots$$

$$FV \text{ of}(n-1)^{\text{th}} A = A[1+k]$$

$$FV \text{ of } n^{th} A = A$$

Thus the future value of this annuity will be

$$\begin{aligned} FVA &= A[1+(n-1)k] + A[1+(n-2)k] + \cdots + A(1+2k) + A(1+k) + A \\ &= A + A + \cdots + A + Ak[(n-1)+(n-2)+\cdots+2+1] \\ &= nA + Ak[(n-1)+(n-2)+\cdots+2+1] \qquad (3.3) \\ &= nA + Ak\left[\frac{n(n-1)}{2}\right] \\ &= nA\left[1+\frac{(n-1)k}{2}\right] \end{aligned}$$

Rule of 69: An amount at annual interest rate k will double in approximately

$$n = \frac{69}{100k} + .35$$

years where100 k is the rate of interest in *percentage.*

An amount will double in n years at an approximate annual rate of interest of From the above rule of 69 you can also obtain k if n is known.

$$k = \frac{100(n-.35)}{69}$$

Note that k is in fractions.

It should be noted that the value of n, obtained using the rule of 69, is underestimated for low interest rates, almost the same for intermediate values of the interest rate, and overestimated for higher values of the interest rate. The following example clarifies this point.

Example 4.6: In how many years will \$100 grow to \$200 at annual rates of interest of .005, .01, .05, .10, .15, .20, .25, .30, .40, and .50?

Answer: Using formula (2.12) and the rule of 69, the exact and approximate values of n for various values of k are obtained as follows:

k	Exact n using equation (2.11)	Approximate n using equation (2.12)	Underestimated (-) or overestimated (+)
.005	138.98	138.35	-0.63
.01	69.66	69.35	-0.31
.05	14.21	14.15	-0.06
.10	7.27	7.25	-0.02
.15	4.95	4.95	0
.20	3.80	3.80	0
.25	3.11	3.11	0
.30	2.64	2.65	+.01
.40	2.06	2.075	+.015
.50	1.71	1.73	+.02

