November 22, 2002

The integral of $\sec x$ is a little tricky:

$$\int \sec x\, dx = \int \frac{\sec x\,(\tan x + \sec x)}{\sec x + \tan x}\, dx = \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\, dx =$$
$$\int \frac{du}{u} = \ln|u| + C = \boxed{\ln|\sec x + \tan x| + C},$$

where $u = \sec x + \tan x$, $du = (\sec x \tan x + \sec^2 x)\, dx$.

Analogously:

$$\int \csc x\, dx = \boxed{-\ln|\csc x + \cot x| + C}.$$

More generally an integral of the form

$$\int \tan^m x \sec^n x\, dx$$

can be computed in the following way:

(1) If $m$ is odd, use $u = \sec x$, $du = \sec x \tan x\, dx$.
(2) If $n$ is even, use $u = \tan x$, $du = \sec^2 x\, dx$.

*Example*: $\displaystyle\int \tan^3 x \sec^2 x\, dx = \cdots$

Since in this case $m$ is odd and $n$ is even it does not matter which method we use, so let's use the first one:

$(u = \sec x,\ du = \sec x \tan x\, dx)$

$$\begin{aligned}
\cdots = \int \underbrace{\tan^2 x}_{u^2-1}\ \underbrace{\sec x}_{u}\ \underbrace{\tan x \sec x\, dx}_{du} &= \int (u^2 - 1) u\, du \\
&= \int (u^3 - u)\, du \\
&= \frac{u^4}{4} - \frac{u^2}{2} + C \\
&= \boxed{\tfrac{1}{4}\sec^4 x - \tfrac{1}{2}\sec^2 x + C}.
\end{aligned}$$

Next let's solve the same problem using the second method:

*Example*:

$$
\begin{aligned}
\int \sqrt{x^2 - 4x + 5}\, dx &= \int \sqrt{(x-2)^2 + 1}\, dx \\
&= \int \sqrt{u^2 + 1}\, du && (u = x - 2) \\
&= \int \sqrt{\tan^2 t + 1} \cdot \sec^2 t\, dt && (u = \tan t) \\
&= \int \sec^3 t\, dt \\
&= \frac{\sec t \tan t}{2} + \frac{1}{2} \ln|\sec t + \tan t| + C \\
&= \frac{u\sqrt{u^2+1}}{2} + \frac{1}{2} \ln|u + \sqrt{u^2+1}| + C \\
&= \frac{(x-2)\sqrt{x^2-4x+5}}{2} \\
&\quad + \frac{1}{2} \ln\left|(x-2) + \sqrt{x^2-4x+5}\right| + C\,.
\end{aligned}
$$

## 1.9. Numerical Integration

Sometimes the integral of a function cannot be expressed with *elementary functions*, i.e., polynomial, trigonometric, exponential, logarithmic, or a suitable combination of these. However, in those cases we still can find an approximate value for the integral of a function on an interval.

### 1.9.1. Trapezoidal Approximation.

A first attempt to approximate the value of an integral $\int_a^b f(x)\,dx$ is to compute its Riemann sum:

$$R = \sum_{i=1}^{n} f(x_i^*)\,\Delta x\,.$$

Where $\Delta x = x_i - x_{i-1} = (b-a)/n$ and $x_i^*$ is some point in the interval $[x_{i-1}, x_i]$. If we choose the left endpoints of each interval, we get the *left-endpoint approximation*:

$$L_n = \sum_{i=1}^{n} f(x_{i-1}))\Delta x = (\Delta x)\{f(x_0) + f(x_1) + \cdots + f(x_{n-1})\}\,,$$

Similarly, by choosing the right endpoints of each interval we get the *right-endpoint approximation*:

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = (\Delta x)\{f(x_1) + f(x_2) + \cdots + f(x_n)\}\,.$$

The *trapezoidal approximation* is the average of $L_n$ and $R_n$:

$$T_n = \frac{1}{2}(L_n + R_n) = \frac{\Delta x}{2}\{f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\}\,.$$

*Example*: Approximate $\int_0^1 x^2\,dx$ with trapezoidal approximation using 4 intervals.

*Solution*: We have $\Delta x = 1/4 = 0.25$. The values for $x_i$ and $f(x_i) = x_i^2$ can be tabulated in the following way:

| $i$ | $x_i$ | $f(x_i)$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0.25 | 0.0625 |
| 2 | 0.5 | 0.25 |
| 3 | 0.75 | 0.5625 |
| 4 | 1 | 1 |

Analogously:

$$[F(x)]_{-\infty}^{a} = \lim_{t\to-\infty} [F(x)]_{t}^{a} \,,$$

and

$$[F(x)]_{-\infty}^{\infty} = [F(x)]_{-\infty}^{c} + [F(x)]_{c}^{\infty} = \lim_{t\to-\infty} [F(x)]_{t}^{c} + \lim_{t\to\infty} [F(x)]_{c}^{t} \,.$$

*Example*:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{t}+1\right) = 1\,,$$

or in simplified notation:

$$\int_1^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^\infty = \lim_{t\to\infty}\left(-\frac{1}{t}+1\right) = 1\,.$$

*Example*: For what values of $p$ is the following integral convergent?:

$$\int_1^\infty \frac{1}{x^p}\,dx\,.$$

*Answer*: If $p=1$ then we have

$$\int_1^t \frac{1}{x}\,dx = [\ln x]_1^t = \ln t\,,$$

so

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty}\ln t = \infty\,,$$

and the integral is divergent. Now suppose $p \neq 1$:

$$\int_1^t \frac{1}{x^p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \frac{1}{1-p}\left\{\frac{1}{t^{p-1}} - 1\right\}$$

If $p>1$ then $p-1>0$ and

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\frac{1}{1-p}\left\{\frac{1}{t^{p-1}} - 1\right\} = 0\,,$$

hence the integral is convergent. On the other hand if $p<1$ then $p-1<0$, $1-p>0$ and

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\frac{1}{1-p}\left\{t^{1-p} - 1\right\} = \infty\,,$$

hence the integral is divergent. So:

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \text{is convergent if } p>1 \text{ and divergent if } p\le 1.$$

## 2.2. Volumes

**2.2.1. Volumes by Slices.** First we study how to find the volume of some solids by the method of cross sections (or "slices"). The idea is to divide the solid into slices perpendicular to a given reference line. The volume of the solid is the sum of the volumes of its slices.

**2.2.2. Volume of Cylinders.** A cylinder is a solid whose cross sections are parallel translations of one another. The volume of a cylinder is the product of its height and the area of its base:

$$V = Ah\,.$$

**2.2.3. Volume by Cross Sections.** Let $R$ be a solid lying alongside some interval $[a,b]$ of the $x$-axis. For each $x$ in $[a,b]$ we denote $A(x)$ the area of the cross section of the solid by a plane perpendicular to the $x$-axis at $x$. We divide the interval into $n$ subintervals $[x_{i-1},x_i]$, of length $\Delta x = (b-a)/n$ each. The planes that are perpendicular to the $x$-axis at the points $x_0, x_1, x_2, \dots, x_n$ divide the solid into $n$ slices. If the cross section of $R$ changes little along a subinterval $[x_{i-1},x_i]$, the slab positioned alongside that subinterval can be considered a cylinder of height $\Delta x$ and whose base equals the cross section $A(x_i^*)$ at some point $x_i^*$ in $[x_{i-1},x_i]$. So the volume of the slice is



$$\Delta V_i \approx A(x_i^*)\,\Delta x\,.$$

The total volume of the solid is

$$V = \sum_{i=1}^{n} \Delta V_i \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x\,.$$

Once again we recognize a Riemann sum at the right. In the limit as $n \to \infty$ we get the so called *Cavalieri's principle*:

$$V = \int_a^b A(x)\,dx\,.$$

Of course, the formula can be applied to any axis. For instance if a solid lies alongside some interval $[a,b]$ on the $y$ axis, the formula becomes

$$V = \int_a^b A(y)\,dy\,.$$

*Example*: Find the volume of a cone of radius $r$ and height $h$.

If the revolution is performed around the $y$-axis, then:

$$V = \int_a^b \pi \left[ (x_R)^2 - (x_L)^2 \right] \, dy \, .$$

*Example*: Find the volume of the solid obtained by revolving the area between $y = x^2$ and $y = \sqrt{x}$ around the $x$-axis.

*Solution*: First we need to find the intersection points of these curves in order to find the interval of integration:

$$\begin{cases} y = x^2 \\ y = \sqrt{x} \end{cases} \quad \Rightarrow \quad (x, y) = (0, 0) \quad \text{and} \quad (x, y) = (1, 1) \, ,$$

hence we must integrate from $x = 0$ to $x = 1$:

$$\begin{aligned} V &= \pi \int_0^1 \left[ (\sqrt{x})^2 - (x^2)^2 \right] \, dx = \pi \int_0^1 \left( x - x^4 \right) \, dx \\ &= \pi \left[ \frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = \pi \left( \frac{1}{2} - \frac{1}{5} \right) = \frac{3\pi}{10} \, . \end{aligned}$$



**2.2.5. Volumes by Shells.** Next we study how to find the volume of some solids by the method of shells. Now the idea is to divide the solid into shells and add up their volumes.

**2.2.6. Volume of a Cylindrical Shell.** A cylindrical shell is the region between two concentric circular cylinders of the same height $h$. If their radii are $r_1$ and $r_2$ respectively, then the volume is:

$$V = \pi r_2^2 h - \pi r_1^2 h = \pi h (r_2^2 - r_1^2) = \pi h \overbrace{(r_2 + r_1)}^{2\overline{r}} \overbrace{(r_2 - r_1)}^{t} = 2\pi \overline{r} t h \, ,$$

where $\overline{r} = (r_2 + r_1)/2$ is the average radius, and $t = r_2 - r_1$ is the thickness of the shell.

**2.2.7. Volumes by Cylindrical Shells.** Consider the solid generated by revolving around the $y$-axis the region under the graph of $y = f(x)$ between $x = a$ and $x = b$. We divide the interval $[a, b]$ into $n$ subintervals $[x_{i-1}, x_i]$ of length $\Delta x = (b - a)/n$ each. The volume $V$ of the solid is the sum of the volumes $\Delta V_i$ of the shells determined by the partition. Each shell, obtained by revolving the region under $y = f(x)$ over the subinterval $[x_{i-1}, x_i]$, is approximately cylindrical. Its height

If the region is revolved around the $x$-axis then the variables $x$ and $y$ reverse their roles:

$$V = \int_a^b 2\pi y \, (x_R - x_L) \, dy \, .$$

**2.2.9. Revolving Around an Arbitrary Line.** If the plane region is revolved around a vertical line $y = c$, the radius of the shell will be $x - c$ (or $c - x$, whichever is positive) instead of $x$, so the formula becomes:

$$V = \int_a^b 2\pi(x - c)(f(x) - g(x)) \, dx = \int_a^b 2\pi(x - c)(y_T - y_B) \, dx \, .$$

Similarly, if the region is revolved around the horizontal line $x = c$, the formula becomes:

$$V = \int_a^b 2\pi(y - c)(f(y) - g(y)) \, dy = \int_a^b 2\pi(y - c)(x_R - x_L) \, dy \, ,$$

where $y - c$ must be replaced by $c - y$ if $c > y$.

*Answer*: The given points correspond to the values $t = 1$ and $t = 2$ of the parameter, so:

$$\begin{aligned}
L &= \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_1^2 \sqrt{(2t)^2 + (3t^2)^2}\, dt \\
&= \int_1^2 \sqrt{4t^2 + 9t^4}\, dt \\
&= \int_1^2 t\sqrt{4 + 9t^2}\, dt \\
&= \frac{1}{18}\int_{13}^{40} \sqrt{u}\, du \qquad\qquad (u = 4 + 9t^2) \\
&= \frac{1}{27}\left[40^{3/2} - 13^{3/2}\right] \\
&= \boxed{\frac{1}{27}(80\sqrt{10} - 13\sqrt{13})}\,.
\end{aligned}$$



In cases when the arc is given by an equation of the form $y = f(x)$ or $x = f(x)$ the formula becomes:

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

or

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy}$$

*Example*: Find the length of the arc defined by the curve $y = x^{3/2}$ between the points $(0, 0)$ and $(1, 1)$.

## 2.4. Average Value of a Function (Mean Value Theorem)

**2.4.1. Average Value of a Function.** The average value of finitely many numbers $y_1, y_2, \dots, y_n$ is defined as

$$y_{\text{ave}} = \frac{y_1 + y_2 + \cdots + y_n}{n}.$$

The average value has the property that if each of the numbers $y_1, y_2, \dots, y_n$ is replaced by $y_{\text{ave}}$, their sum remains the same:

$$y_1 + y_2 + \cdots + y_n = \overbrace{y_{\text{ave}} + y_{\text{ave}} + \cdots + y_{\text{ave}}}^{(n \text{ times})}$$

Analogously, the average value of a function $y = f(x)$ in the interval $[a, b]$ can be defined as the value of a constant $f_{\text{ave}}$ whose integral over $[a, b]$ equals the integral of $f(x)$:

$$\int_a^b f(x)\,dx = \int_a^b f_{\text{ave}}\,dx = (b-a)\,f_{\text{ave}}.$$

Hence:

$$\boxed{f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx}.$$



**2.4.2. The Mean Value Theorem for Integrals.** If $f$ is continuous on $[a, b]$, then there exists a number $c$ in $[a, b]$ such that

$$f(c) = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

i.e.,

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

*Example*: Assume that in a certain city the temperature (in °F) $t$ hours after 9 A.M. is represented by the function

$$T(t) = 50 + 14\sin\frac{\pi t}{12}.$$

Find the average temperature in that city during the period from 9 A.M. to 9 P.M.

*Answer*:

$$
\begin{aligned}
T_{\text{ave}} &= \frac{1}{12-0}\int_0^{12}\left(50+14\sin\frac{\pi t}{12}\right)\,dt \\
&= \frac{1}{12}\left[50t-\frac{14\cdot 12}{\pi}\cos\frac{\pi t}{12}\right]_0^{12} \\
&= \frac{1}{12}\left\{\left(50\cdot 12-\frac{168}{\pi}\cos\frac{12\pi}{12}\right)-\left(50\cdot 0-\frac{168}{\pi}\cos 0\right)\right\} \\
&= 50+\frac{28}{\pi}\approx 58.9\,.
\end{aligned}
$$

hand $f(x) = 0$ for $x$ outside $[2,5]$, hence:

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{2}^{5} c\,dx = c(5-2) = 3c\,,$$

so $c = 1/3$. Hence

$$f(x) = \begin{cases} 1/3 & \text{if } 2 \le x \le 5, \\ 0 & \text{otherwise.} \end{cases}$$

**2.6.2. Means.** The *mean* or *average* of a discrete random variable that takes values $x_1, x_2, \dots, x_n$ with probabilities $p_1, p_2, \dots, p_n$ respectively is

$$\overline{x} = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n = \sum_{i=1}^{n} x_i p_i\,.$$

For instance the mean value of the points obtained by rolling a dice is

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{7}{2} = 3.5\,.$$

This means that if we roll the dice many times in average we may expect to get about 3.5 points per roll.



For continuous random variables the probability is replaced with the probability density function, and the sum becomes an integral:

$$\mu = \overline{x} = \int_{-\infty}^{\infty} x f(x)\,dx\,.$$

**2.6.3. Waiting Times.** The time that we must wait for some event to occur (such as receiving a telephone call) can be modeled with a random variable of density

$$f(t) = \begin{cases} 0 & \text{if } t < 0, \\ ce^{-ct} & \text{if } t \ge 0, \end{cases}$$

were $c$ is a positive constant. Note that, as expected:

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_{0}^{\infty} ce^{-ct}\,dx = \left[-e^{-ct}\right]_0^{\infty} = \lim_{u\to\infty} \{-e^{-cu} - (-e^{0})\} = 1\,.$$

$$\begin{aligned}
P(70 \leq X \leq 130) &= \int_{70}^{130} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450}\, dx \\
&= \frac{1}{\sqrt{\pi}} \int_{-\sqrt{2}}^{\sqrt{2}} e^{-u^2}\, du && [u = (x-100)/15\sqrt{2}] \\
&= \frac{2}{\sqrt{\pi}} \int_{0}^{\sqrt{2}} e^{-u^2}\, du && \text{(by symmetry)} \\
&= \phi(\sqrt{2}) \approx \phi(1.4) \approx \boxed{0.952}
\end{aligned}$$

**3.1.5. Separable Differential Equations.** A differential equation is said to be *separable* if it can be written in the form

$$f(y)\,dy = g(x)\,dx\,,$$

so that the left hand side depends on $y$ only and the right hand side depends on $x$ only. In particular this is true if the equation is of the form

$$\frac{dy}{dx} = g(x)\,\phi(y)\,,$$

where the right hand side is a product of a function of $x$ and a function of $y$. In this case we get:

$$\frac{1}{\phi(y)}\,dy = g(x)\,dx\,.$$

Given the equation

$$f(y)\,dy = g(x)\,dx\,,$$

we can solve it by integrating both sides. Since the antiderivatives of a function differ in a constant, we get:

$$\int f(y)\,dy = \int g(x)\,dx + C\,,$$



If $F(y) = \int f(y)\,dy$ and $G(x) = \int g(x)\,dx$ then the solution takes the form

$$F(y) = G(x) + C\,.$$

Next we will try to solve this equation algebraically in order to either write $y$ as a function of $x$, or $x$ as a function of $y$.

*Example*: Consider the equation

$$\frac{dy}{dx} = y^2\,x\,.$$

The right hand side is the product of a function of $x$ and a function of $y$, so it is separable:

$$\frac{1}{y^2}\,dy = x\,dx\,.$$

Integrating both sides we get:

$$-\frac{1}{y} = \frac{x^2}{2} + C\,,$$

hence

$$y = -\frac{2}{x^2 + 2C} = -\frac{2}{x^2 + C'}\,,$$

where $C'$ is a new constant equal to $2C$.

**3.1.6. Initial Value Problems.** A differential equation together with an initial condition

$$\begin{cases} \dfrac{dy}{dx} = F(x, y) \\ y(x_0) = y_0 \end{cases}$$

is called an *initial value problem.*

The initial condition can be used to determine the value of the constant in the solution of the equation.

*Example*: Solve the following initial value problem:

$$\begin{cases} \dfrac{dy}{dx} = y^2 x \\ y(0) = 1 \end{cases}$$

*Solution*: We already found the general solution to the differential equation:

$$y = -\frac{2}{x^2 + C}$$

Next we let $x = 0$ and $y = 1$, and solve for $C$:

$$1 = -\frac{2}{C} \implies C = -2 .$$

So the solution is

$$y = -\frac{2}{x^2 - 2} = \frac{2}{2 - x^2} .$$

**4.1.3. Operations with Limits.** If $a_n \to a$ and $b_n \to b$ then:

$(a_n + b_n) \to a + b$.

$(a_n - b_n) \to a - b$.

$ca_n \to ca$ for any constant $c$.

$a_n b_n \to ab$.

$\dfrac{a_n}{b_n} \to \dfrac{a}{b}$ if $b \neq 0$.

$(a_n)^p \to a^p$ if $p > 0$ and $a_n > 0$ for every $n$.

*Example*: Find $\displaystyle\lim_{n\to\infty} \frac{n^2+n+1}{2n^2+3}$.

*Answer*: We divide by $n^2$ on top and bottom and operate with limits inside the expression:

$$\lim_{n\to\infty} \frac{n^2+n+1}{2n^2+3} = \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n^2}}{2+\frac{3}{n^2}} = \frac{1+0+0}{2+0} = \boxed{\frac{1}{2}}.$$



**4.1.4. Squeeze Theorem.** If $a_n \le b_n \le c_n$ for every $n \ge n_0$ and $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\displaystyle\lim_{n\to\infty} b_n = L$.

Consequence: If $\displaystyle\lim_{n\to\infty} |a_n| = 0$ then $\displaystyle\lim_{n\to\infty} a_n = 0$.

*Example*: Find $\displaystyle\lim_{n\to\infty} \frac{\cos n}{n}$.

*Answer*: We have $-\dfrac{1}{n} \le \dfrac{\cos n}{n} \le \dfrac{1}{n}$, and $\dfrac{1}{n} \to 0$ as $n \to \infty$, hence by the squeeze theorem

$$\boxed{\lim_{n\to\infty} \frac{\cos n}{n} = 0}.$$

**4.1.5. Other definitions.**

4.1.5.1. *Increasing, Decreasing, Monotonic.* A sequence is *increasing* if $a_{n+1} > a_n$ for every $n$. It is *decreasing* if $a_{n+1} < a_n$ for every $n$. It is called *monotonic* if it is either increasing or decreasing.

*Example*: Probe that the sequence $a_n = \dfrac{n+1}{n}$ is decreasing.

Now we prove that $a_n$ is increasing:

$$(a_{n+1})^2 = 2 + a_n > a_n + a_n = 2a_n > a_n \cdot a_n = (a_n)^2\,,$$

hence $a_{n+1} > a_n$.

Finally, since the given sequence is bounded and increasing, by the monotonic sequence theorem it has a limit $L$. We can find it by taking limits in the recursive relation:

$$a_{n+1} = \sqrt{2 + a_n}\,.$$

Since $a_n \to L$ and $a_{n+1} \to L$ we have:

$$L = \sqrt{2+L} \quad \Rightarrow \quad L^2 = 2 + L \quad \Rightarrow L^2 - L - 2 = 0\,.$$

That equation has two solutions, $-1$ and $2$, but since the sequence is positive the limit cannot be negative, hence $L = 2$.

Note that the trick works only when we know for sure that the limit exists. For instance if we try to use the same trick with the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$ ($f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$), calling $L$ the "limit" we get from the recursive relation that $L = L + L$, hence $L = 0$, so we "deduce" $\lim_{n\to\infty} f_n = 0$. But this is wrong, in fact the Fibonacci sequence is divergent.

*Example*: Show that $\displaystyle\sum_{n=1}^{\infty} \sin n$ diverges.

*Answer*: All we need to show is that $\sin n$ does not tend to 0. If for some value of $n$, $\sin n \approx 0$, then $n \approx k\pi$ for some integer $k$, but then

$$\begin{aligned}\sin(n+1) &= \sin n \cos 1 + \cos n \sin 1\\ &\approx \sin k\pi \cos 1 + \cos k\pi \sin 1\\ &= 0 \pm \sin 1\\ &= \pm 0.84\cdots \neq 0\end{aligned}$$

So if a term $\sin n$ is close to zero, the next term $\sin(n+1)$ will be far from zero, so it is impossible for $\sin n$ to get permanently closer and closer to 0.

**4.2.6. Operations with Series.** If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series and $c$ is a constant then the following series are also convergent and:

(1) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(2) $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(3) $\displaystyle\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

From here we get a system of $n+1$ equations with the following solution:

$$\begin{aligned} c_0 &= f(a) \\ c_1 &= f'(a) \\ c_2 &= \frac{f''(a)}{2!} \\ &\dots \\ c_n &= \frac{f^{(n)}(a)}{n!} \end{aligned}$$

hence:

$$\begin{aligned} T_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \,. \end{aligned}$$

That polynomial is the so called *nth-degree Taylor polynomial of $f(x)$ at $x = a$.*

*Example*: The third-degree Taylor polynomial of $f(x) = \sin x$ at $x = a$ is

$$T_3(x) = \sin a + \cos a \cdot (x-a)^2 - \frac{\sin a}{2}(x-a)^2 - \frac{\cos a}{3!}(x-a)^3 \,.$$



For $a = 0$ we have $\sin 0 = 0$ and $\cos 0 = 1$, hence:

$$T_3(x) = x - \frac{x^3}{6} \,.$$

So in particular

$$\sin 0.1 \approx 0.1 - \frac{0.1^3}{6} = 0.09983333\ldots \,.$$

The actual value of $\sin 0.1$ is

$$\sin 0.1 = 0.099833416 \,,$$

which agrees with the value obtained from the Taylor polynomial up to the sixth decimal place.

**4.7.2. Taylor's Inequality.** The difference between the value of a function and its Taylor approximation is called *remainder*:

$$R_n(x) = f(x) - T_n(x) \,.$$

$$
\begin{aligned}
y'' - 2xy' + y &= \sum_{n=0}^{\infty} c_n x^n - x\sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \\
&= \sum_{n=0}^{\infty} c_n x^n - \sum_{n=1}^{\infty} n c_n x^{n} + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}
\end{aligned}
$$

After some reindexing and grouping we get that the equation becomes:

$$\sum_{n=0}^{\infty} \{(n+2)(n+1)c_{n+2} - (2n-1)c_n\}\, x^n = 0\,,$$

which implies:

$$c_{n+2} = \frac{2n-1}{(n+1)(n+2)}\, c_n\,.$$

The first two coefficients $c_0$ and $c_1$ are arbitrary, and the rest can be computed using that relation:

$$
\begin{aligned}
c_2 &= \frac{-1}{2} c_0 \\
c_3 &= \frac{1}{2\cdot 3} c_0 \\
c_4 &= \frac{3}{3\cdot 4} c_2 = -\frac{3}{4!} c_0 \\
c_5 &= \frac{5}{4\cdot 5} c_3 = \frac{5}{5!} c_0 \\
&\cdots
\end{aligned}
$$



In general the even and odd coefficients are:

$$
\begin{aligned}
c_{2n} &= \frac{(-1)\cdot 3\cdot 7\cdot 11\cdot\cdots\cdot(4n-5)}{(2n)!}\, c_0 \\
c_{2n+1} &= \frac{1\cdot 5\cdot 9\cdot\cdots\cdot(4n-3)}{(2n+1)!}\, c_1\,,
\end{aligned}
$$

and the solution is

$$
\begin{aligned}
y = c_0 &\left\{1 + \sum_{n=1}^{\infty} \frac{(-1)\cdot 3\cdot 7\cdot 11\cdot\cdots\cdot(4n-5)}{(2n)!}\, x^{2n}\right\} \\
&\qquad + c_1\left\{x + \sum_{n=1}^{\infty} \frac{1\cdot 5\cdot 9\cdot\cdots\cdot(4n-3)}{(2n+1)!}\, x^{2n+1}\right\}
\end{aligned}
$$