(D) DIFFERENTIATING LOGARITHMIC FUNCTIONS

If $y = \log_e x$ (or $y = \ln x$) then $e^y = x$

Since $x = e^y$, then $\frac{dx}{dy} = e^y$

As $\frac{dy}{dx} = \frac{1}{dx/dy}$, we can say $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$

If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$

If $y = \ln f(x)$, then $\frac{dy}{dx}$: $\boxed{\frac{d}{dx}\{\ln f(x)\} = \frac{1}{f(x)} \times f'(x) = \frac{f'(x)}{f(x)}}$

Note:

$y = a\ln x$, $\frac{dy}{dx} = \frac{a}{x}$

$y = \ln(ax)$, $\frac{dy}{dx} = \frac{1}{x}$

$y = \ln(ax+b)$, $\frac{dy}{dx} = \frac{a}{ax+b}$

Example:

① Find $\frac{dy}{dx}$:

(a) $y = \ln 5x$

$\frac{dy}{dx} = \frac{1}{5x} \times 5$

$= \frac{1}{x}$

(b) $y = \ln(3x^2 - 2)$

$= \frac{1}{3x^2-2} \times 6x$

$= \frac{6x}{3x^2-2}$

(c) $y = \ln(e^x - 3x)$

$\frac{dy}{dx} = \frac{1}{e^x - 3x} \times (e^x - 3)$

$= \frac{e^x - 3}{e^x - 3x}$

(d) $y = \ln(5x-1)^7$

$= 7\ln(5x-1)$

$\frac{dy}{dx} = 7 \times \frac{1}{5x-1} \times 5$

$= \frac{35}{5x-1}$

② Find derivative of the following functions:

(a) $y = \underbrace{\sqrt{x}}_{u}\ \underbrace{\ln x}_{v}$ ← product rule

$\sqrt{x} = x^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}$

$a^m \times a^n = a^{m+n}$

$\frac{dy}{dx} = \sqrt{x}\left(\frac{1}{x}\right) + \ln(x)\left(\frac{1}{2}x^{-1/2}\right)$

$= (x^{1/2})(x^{-1}) + \ln x\left(\frac{1}{2}x^{-1/2}\right)$

$= x^{-1/2} + \frac{x^{-1/2}}{2}\ln x$

$= x^{-1/2}\left[1 + \frac{1}{2}\ln x\right]$

$= \frac{1}{\sqrt{x}}\left(1 + \frac{1}{2}\ln x\right)$

(b) $y = \overset{u}{2e^x}\ \overset{v}{\ln 2x}$

$\frac{dy}{dx} = 2e^x\left(\frac{1}{2x} \times 2\right) + \ln 2x(2e^x)$

$= 2e^x\left[\frac{1}{x} + \ln 2x\right]$

(c) $y = \frac{7\ln x - x^3}{e^{3x}}$ ← Quotient rule

$\frac{dy}{dx} = \frac{e^{3x}\left(\frac{7}{x} - 3x^2\right) - (7\ln x - x^3)(3e^{3x})}{(e^{3x})^2}$

$= \frac{e^{3x}\left[\frac{7}{x} - 3x^2 - 21\ln x + 3x^3\right]}{(e^{3x})^2}$

$= \frac{\frac{7}{x} - 3x^2 - 21\ln x + 3x^3}{e^{3x}}$

③ Find exact value of gradient to curve $y = e^x - \frac{\ln x}{2}$ when $x = \ln 3$

$y = e^x - \frac{\ln x}{2}$

$\frac{dy}{dx} = e^x - \frac{1}{2}\left(\frac{1}{x}\right)$

$= e^x - \frac{1}{2x}$

when $x = \ln 3$, $\frac{dy}{dx} = e^{\ln 3} - \frac{1}{2(\ln 3)}$

$= 3 - \frac{1}{2\ln 3}$ or $3 - \frac{1}{\ln 9}$

(E) DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS

$y = \sin x$, $\frac{dy}{dx} = \cos x$	$y = \cos x$, $\frac{dy}{dx} = -\sin x$	$y = \tan x$, $\frac{dy}{dx} = \sec^2 x$
$y = \sin(ax+b)$, $\frac{dy}{dx} = a\cos(ax+b)$	$y = \cos(ax+b)$, $\frac{dy}{dx} = -a\sin(ax+b)$	$y = \tan(ax+b)$, $\frac{dy}{dx} = a\sec^2(ax+b)$
$y = \sin f(x)$, $\frac{dy}{dx} = f'(x)\cos f(x)$	$y = \cos f(x)$, $\frac{dy}{dx} = -f'(x)\sin f(x)$	$y = \tan f(x)$, $\frac{dy}{dx} = f'(x)\sec^2 f(x)$

EXPLANATION:

$y = \tan x$

$y = \frac{\sin x}{\cos x}$

$\frac{dy}{dx} = \frac{\cos x(\cos x) - \sin x(\sin x)}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

$= \left(\frac{1}{\cos x}\right)^2$

$= \sec^2 x$

Example: Find $\frac{dy}{dx}$

(a) $y = \sin 4x$

$\frac{dy}{dx} = 4\cos 4x$

(b) $y = \cos 3x$

$\frac{dy}{dx} = -3\sin x$

(c) $y = \sin(x^2+2)$

$\frac{dy}{dx} = 2x\cos(2x^2+2$

(d) $y = 5\sin 3x^2$

$\frac{dy}{dx} = 5 \times (6x\cos 3x^2)$

$= 30x\cos 3x^2$

(e) $y = 7\tan\left(4x + \frac{\pi}{2}\right)$

$\frac{dy}{dx} = 7 \times \left(4\sec^2\left(4x + \frac{\pi}{2}\right)\right)$

$= 28\sec^2\left(4x + \frac{\pi}{2}\right)$

If $y = \sin^n x$, then $\frac{dy}{dx} = n\sin^{n-1}x\cos x$
If $y = \cos^n x$, then $\frac{dy}{dx} = -n\cos^{n-1}x\sin x$
} chain rule

EXPLANATION:

$y = \sin^n x$

Let $u = \sin x$ $y = u^n$

$\frac{du}{dx} = \cos x$ $\frac{dy}{du} = nu^{n-1}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= nu^{n-1}(\cos x)$

$= n\sin^{n-1}x\cos x$

example: Find $\frac{dy}{dx}$ a → c (basic)
d (quotient / product rule)

(a) $y = \cos^2 x$

$\frac{dy}{dx} = 2\cos x(-\sin x)$

$= -2\sin x\cos x$

$= -\sin 2x$

(b) $y = (\sin x + \cos x)^5$

$\frac{dy}{dx} = 5(\sin x + \cos x)^4(\cos x - \sin x)$

$= 5(\cos x - \sin x)(\sin x + \cos x)^4$

(c) $y = (3 - 2\cos 5x)^4$

$\frac{dy}{dx} = 4(3 - 2\cos 5x)^3(-2(-5\sin 5x))$

$= 40\sin 5x(3 - 2\cos 5x)^3$

(d) $y = \overset{u}{x^2}\ \overset{v}{\sin x}$

$\frac{dy}{dx} = x^2(\cos x) + \sin x(2x)$

$= x^2\cos x + 2x\sin x$

$= x[x\cos x + 2\sin x]$

(21) Find coordinates of turning points on the curve $y = 2\sin x + \cos 2x$ for $0 < x < \pi$ and determine whether these points are max/min pts.

$\frac{dy}{dx} = 2\cos x - 2\sin 2x$

$\frac{d^2y}{dx^2} = -2\sin x - 2(2\cos 2x)$

$= -2\sin x - 4\cos 2x$

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Turning pt: $\frac{dy}{dx} = 0$

$2\cos x - 2\sin 2x = 0$

$2\cos x - 2(2\sin x\cos x) = 0$

$2\cos x - 4\sin x\cos x = 0$

$2\cos x(1 - 2\sin x) = 0$

$\cos x = 0$ OR $\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}$, $\frac{3\pi}{2}$ (rejected) $\alpha = \frac{\pi}{6} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\left(\frac{\pi}{2}, 1\right)$ $x = \frac{\pi}{2}$, $y = 1$ $\frac{d^2y}{dx^2} = 2 > 0$ ∴ min pt

$\left(\frac{\pi}{6}, \frac{3}{2}\right)$ $x = \frac{\pi}{6}$, $y = \frac{3}{2}$, $\frac{d^2y}{dx^2} = -3 < 0$ ∴ max pt

$\left(\frac{5\pi}{6}, \frac{3}{2}\right)$ $x = \frac{5\pi}{6}$, $y = \frac{3}{2}$, $\frac{d^2y}{dx^2} = -3 < 0$ ∴ max pt