CHAPTER 8: Integration II

(A) INTEGRATION OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

$$\int e^x = e^x + c$$
$$\int e^{ax+b} = \frac{e^{ax+b}}{a} + c$$

$y = e^{ax+b}$

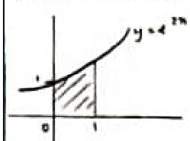
$\frac{dy}{dx} = a e^{ax+b}$

$\int a e^{ax+b} = e^{ax+b} + c$

Example:

(a) $\int e^{3x+2}\,dx = \frac{e^{3x+2}}{3} + c$, (b) $\int_0^1 e^{2x+3}\,dx = \left[\frac{e^{2x+3}}{2}\right]_0^1 = 64.2$ (3s.f)

① (a) Find area bounded by graph of $y = e^{2x}$, the x-axis, y-axis and line $x=1$

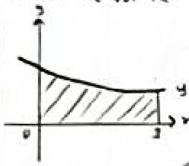


$A = \int_0^1 y\,dx = \int_0^1 e^{2x} = \left[\frac{e^{2x}}{2}\right]_0^1 = \frac{1}{2}\left[e^{2x}\right]_0^1$

$= \frac{1}{2}\left(e^{2(1)} - e^{2(0)}\right)$

$= 3.19$ (3s.f) unit²

(b) The line $y = e^{-x}$ between $x=0$ and $x=3$ is rotated through 360° about the x-axis. Find the volume of solid obtained.



volume generated, $V = \int_0^3 \pi y^2\,dx = \pi\int_0^3 (e^{-x})^2 = \pi\int_0^3 (e^{-2x})\,dx$

$= \pi\left[\frac{e^{-2x}}{-2}\right]_0^3$

$= 1.57$ (3s.f) unit³

because if $\frac{1}{x} = x^{-1}$ → $\int x^{-1}\,dx = \frac{x^{0}}{0}$ ✗ (incorrect)

$$\int \frac{1}{x} = \ln|x| + c$$
$$\int\left(\frac{1}{ax+b}\right) = \frac{1}{a}\ln|ax+b| + c$$
$$\int \frac{f'(x)}{f(x)} = \ln|f(x)| + c$$

$\int \frac{a}{ax+b}\,dx = \ln|ax+b| + c$

$a\int \frac{1}{ax+b}\,dx = \ln|ax+b| + c$

$\therefore \int \frac{1}{ax+b}\,dx = \frac{\ln|ax+b|}{a} + c$

$= \frac{1}{a}\ln|ax+b| + c$

$y = \ln x$, $\frac{dy}{dx} = \frac{1}{x}$

$y = \ln(ax+b)$, $\frac{dy}{dx} = \frac{1}{ax+b} \times a = \frac{a}{ax+b}$

$y = \ln f(x)$, $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

In general:

① $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$

② $\int \frac{1}{(ax+b)^2}\,dx = \int (ax+b)^{-2}\,dx$

$= \frac{(ax+b)^{-1}}{(-1)(a)} + c$

$= -\frac{1}{a(ax+b)} + c$

Better formula: constant

$\int \frac{2}{2x-3}\,dx = \int \frac{2}{2} \times \frac{1}{2x-3}$ differentiation

$= \ln|2x-3| + c$

$\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

Example:

(a) $\int \frac{2}{2x-3}\,dx = \int \frac{2}{2} \cdot \frac{1}{2x-3}$

$= \ln|2x-3| + c$

(b) $\int \frac{2x}{x^2+3}\,dx = \ln|x^2+3| + c$ (constant, diff)

(c) $\int \frac{e^{2x}}{e^{2x}+3}\,dx = \int \frac{e^{2x}}{2e^{2x}} \cdot \frac{1}{e^{2x}+3}$

$= \int \frac{1}{2}\left(\frac{1}{e^{2x}+3}\right)$

$= \frac{1}{2}\int \frac{1}{e^{2x}+3}\,dx$

$= \frac{1}{2}\ln|e^{2x}+3| + c$

(B) INTEGRATION OF TRIGONOMETRIC FUNCTIONS

$\int \sin x\,dx = -\cos x + c$ $\quad$ $\int \cos x\,dx = \sin x + c$

$\int \sin(ax+b)\,dx = \frac{-\cos(ax+b)}{a} + c$ $\quad$ $\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} + c$

$\int \sec^2 x\,dx = \tan x + c$

$\int \sec^2(ax+b)\,dx = \frac{\tan(ax+b)}{a} + c$

Example:

(a) $\int \cos 7x\,dx = \frac{\sin 7x}{7} + c$

(b) $\int \tan 3x\,dx = \int \frac{\sin 3x}{\cos 3x}\,dx$

$= \int \frac{\sin 3x}{-3\sin 3x} \cdot \frac{1}{\cos 3x}\,dx$

$= -\frac{1}{3}\int \frac{1}{\cos 3x}\,dx = -\frac{1}{3}\ln|\cos 3x| + c$

(c) $\int 6\sin^2 x\,dx = 6\int \frac{1}{2}(1 - \cos 2x)\,dx$

$= 3\left[x - \frac{\sin 2x}{2}\right] + c$

$= 3x - \frac{3}{2}\sin 2x + c$

(d) $\int \cos^2 x\,dx = \int \frac{1}{2}(\cos 2x + 1)\,dx$

$= \frac{1}{2}\left[\frac{\sin 2x}{2} + x\right]\,dx$

$= \frac{1}{2}\left(\frac{\sin 2x}{2} + x\right) + c$

$= \frac{1}{4}\sin 2x + \frac{1}{2}x + c$

(C) Integration By Inspection

You should know by now that, $y = \sin^3 x$ then $\frac{dy}{dx} = 3\sin^2 x\cos x$, therefore if we were to integrate $\int \sin^2 x\cos x$, we should guess the answer is $\sin^3 x$ (by looking at the f(x) with power and add power by 1). Here is the example of how this is done:

$\int \sin^2 x\cos x\,dx = \frac{1}{3}\sin^3 x + c$

$y = \sin^3 x$, $\frac{dy}{dx} = 3\sin^2 x\cos x$

$\int 3\sin^2 x\cos x = \sin^3 x + c$

$\int \sin^2 x\cos x = \frac{1}{3}\sin^3 x + c$

Example:

$\int 6\cos x\sin^3 x\,dx = 6\int \cos x\sin^3 x\,dx$

$y = \sin^4 x$

$\frac{dy}{dx} = 4\sin^3 x\cos x$

$= 6\left[\frac{1}{4}\sin^4 x + c\right]$

$= \frac{3}{2}\sin^4 x + c$

$\int 4\sin^3 x\cos x = \sin^4 x + c$

$\int \sin^3 x\cos x = \frac{1}{4}\sin^4 x + c$

check:

$y = \frac{3}{2}\sin^4 x + c$

$\frac{dy}{dx} = \frac{3}{2} \times 4\sin^3 x\cos x$

$= 6\sin^3 x\cos x$

(D) INTEGRATION USING PARTIAL FRACTIONS

Example: $\int \frac{x+3}{(x-2)(x+1)}\,dx$: $\frac{x+3}{(x-2)(x+1)} = \left[\frac{A}{(x-2)} + \frac{B}{(x+1)}\right] \times (x-2)(x+1)$

$F(-1)$: $-1+3 = A(-1+1) + B(-1-2)$ $\quad$ $x+3 = A(x+1) + B(x-2)$

$2 = B(-3)$

$\therefore B = -\frac{2}{3}$

$\therefore \frac{5/3}{x-2} + \frac{-2/3}{x+1} = \frac{5}{3(x-2)} - \frac{2}{3(x+1)}$

$F(2)$: $2+3 = A(2+1) + B(2-2)$

$5 = A(3)$

$\therefore A = \frac{5}{3}$

Integration:

$\int \frac{x+3}{(x-2)(x+1)}\,dx = \int\left(\frac{5}{3(x-2)} - \frac{2}{3(x+1)}\right)dx$

$= \frac{5}{3}\ln|x-2| - \frac{2}{3}\ln|x+1| + c$

$= \frac{1}{3}(5\ln|x-2| - 2\ln|x+1|) + c$

$= \frac{1}{3}\ln\left|\frac{(x-2)^5}{(x+1)^2}\right| + c$

(E) INTEGRATION BY PARTS → Enables us to integrate certain products.

If u and v are functions, then $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ ... (PRODUCT RULE)

Integrate both sides gives:

$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$

$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Integration by parts (for indefinite integrals)

$$\int_a^b u\frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b v\frac{du}{dx}\,dx$$

Integration by parts (for definite integrals)

Example ①: simpler

$\int (x+4)e^x\,dx$

$u = x+4$, $\frac{dv}{dx} = e^x$

$\frac{du}{dx} = 1$, $v = e^x$

$\int (x+4)e^x\,dx = e^x(x+4) - \int e^x\,dx$

$= e^x(x+4) - e^x + c$

if like, takes too long

$u = e^x$, $\frac{dv}{dx} = x+4$

$\frac{du}{dx} = e^x$, $v = \frac{x^2}{2} + 4x$

$= e^x\left(\frac{x^2}{2} + 4x\right) - \int e^x\left(\frac{x^2}{2} + 4x\right)dx$

Example:

(a) $\int x\cos 3x\,dx$

$u = x$, $\frac{dv}{dx} = \cos 3x$

$\frac{du}{dx} = 1$, $v = \frac{\sin 3x}{3}$

$\int x\cos 3x\,dx = \frac{x\sin 3x}{3} - \frac{1}{3}\int \sin 3x\,dx$

$= \frac{x\sin 3x}{3} - \frac{1}{3}\left(\frac{-\cos 3x}{3}\right) + c$

$= \frac{x\sin 3x}{3} + \frac{\cos 3x}{9} + c$

(b) $\int \ln x\,dx \Rightarrow \int 1\ln x\,dx$

$u = \ln x$, $\frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x}$, $v = x$

$\int \ln x\,dx = x\ln x - \int\left(\frac{1}{x} \times x\right)dx$

$= x\ln x - \int 1\,dx$

$= x\ln x - x + c$

(c) $\int 4xe^{2x}\,dx$

$u = 4x$, $\frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 4$, $v = \frac{1}{2}e^{2x}$

$\int 4xe^{2x}\,dx = 4x\left(\frac{1}{2}e^{2x}\right) - \int\left(\frac{e^{2x}}{2} \times 4\right)dx$

$= 2xe^{2x} - \int\left(\frac{4e^{2x}}{2}\right)$

$= 2xe^{2x} - \int(2e^{2x})\,dx$

$= 2xe^{2x} - e^{2x} + c$

(F) INTEGRATION BY SUBSTITUTION

STEP ①: Differentiate the substitution

STEP ②: substitute dx into the integral

Note: when dealing with definite integrals, make sure you express the limits in terms of the new variables.

Example:

(a) Find $\int \frac{1}{x}\ln x$ using the substitution $u = \ln x$

$\int \left(\frac{1}{x}\ln x\,dx\right) = \int u\,du$

$u = \ln x$ $\quad$ $= \frac{u^2}{2} + c$

$\frac{du}{dx} = \frac{1}{x}$ $\quad$ $= \frac{(\ln x)^2}{2} + c$

$du = \frac{1}{x}\,dx$

(b) Find $\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$ using the substitution $u = \tan^{-1}x$

$u = \tan^{-1}x \Rightarrow \tan u = x$

$\frac{du}{dx} = \frac{1}{1+x^2}$

$du = \frac{1}{1+x^2}\,dx$

$\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx = \int e^u(1 + \tan u + \tan^2 u)\,du$

$= \int e^u(\tan u + \sec^2 u)\,du$

$= e^u\tan u + c$

$= xe^{\tan^{-1}x} + c$

By inspection:

$y = xe^{\tan^{-1}x}$

$\frac{dy}{dx} = x\left[\frac{1}{1+x^2}e^{\tan^{-1}x}\right] + e^{\tan^{-1}x}(1)$

$= e^{\tan^{-1}x}\left[\frac{x}{1+x^2} + 1\right]$

$= e^{\tan^{-1}x}\left[\frac{x + 1 + x^2}{1+x^2}\right] = e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)$