(E) Harmonic Form

Expressions of the form $a\cos\theta + b\sin\theta$ arise in many practical situations. The expressions ($a\cos\theta + b\sin\theta = c$) cannot be solved and hence they have to be expressed in the form:

$f(\theta) = a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$

or

$f(\theta) = a\sin\theta \pm b\cos\theta = R\sin(\theta \pm \alpha)$

where $R = \sqrt{a^2+b^2}$ and $R\cos\alpha = a$, $R\sin\alpha = b$

where $\tan\alpha = \frac{b}{a}$ with $0 < a < \frac{\pi}{2}$

Example:

(a) Express $7\sin\theta - 24\cos\theta$ in the form $R\sin(\theta - \alpha)$

$7\sin\theta - 24\cos\theta \equiv R\sin(\theta - \alpha)$

$= R[\sin\theta\cos\alpha - \cos\theta\sin\alpha]$

$7\sin\theta - 24\cos\theta = R\sin\theta\cos\alpha - R\cos\theta\sin\theta$

$R\cos\alpha = 7$ ..(1)

$R\sin\alpha = 24$ ..(2)

$R = \sqrt{7^2+24^2} = 25$

(2) ÷ (1)

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{24}{7}$

$\tan\alpha = \frac{24}{7}$

$\therefore \alpha = \tan^{-1}\left(\frac{24}{7}\right)$

$= 73.74°$ (2 d.p)

$\therefore 7\sin\theta - 24\cos\theta = 25\sin\theta(\theta - 73.74°)$

(b) Solve equation $5\sin 2\theta + 12\cos 2\theta = 11$, $0° < \theta < 180°$

$5\sin 2\theta + 12\cos 2\theta = 11$

$13\sin(2\theta + 67.38°) = 11$

$\sin(2\theta + 67.38°) = \frac{11}{13}$

$\alpha = \sin^{-1}\left(\frac{11}{13}\right)$

$= 57.80°$

$0° < \theta < 180°$

$0° < 2\theta < 360°$

$67.38° < 2\theta + 67.38° < 427.38°$

$2\theta + 67.38° = \cancel{57.80°}, 122.20°, 417.8°$

$\therefore \theta = 27.41°, 175.21°$

$= 27.4°, 175.2°$ (1 d.p)