The following examples illustrate the solution of a differential equation which has been derived from a 'growth' situation.

Example 13

The spread of disease occurs at a rate proportional to the product of the number of people infected and the number not infected. Initially 50 out of a population of 1050 are infected and it is spreading at a rate of 10 new cases per day.

(a) If n is the number infected after t days, show that $\frac{dn}{dt} = \frac{n(1050-n)}{5000}$

(b) Solve this differential equation to find the number of people infected after t days.

(c) How long will it take for 250 people to be infected?

(a) $\frac{dn}{dt} \propto n(1050-n)$

$\frac{dn}{dt} = kn(1050-n)$

when $t=0, n=50, \frac{dn}{dt}=10$

$10 = k(50)(1000)$

$\therefore k = \frac{1}{5000}$

$\therefore \frac{dn}{dt} = \frac{n(1050-n)}{5000}$ (shown)

(b) $\frac{dn}{dt} = \frac{n(1050-n)}{5000}$

$\int \frac{1}{n(1050-n)}\,dn = \int \frac{1}{5000}\,dt$

not log, but partial fraction (distinct linear): $\frac{A}{n} + \frac{B}{1050-n}$

$\frac{1}{n(1050-n)} = \left[\frac{A}{n} + \frac{B}{1050-n}\right] \times n(1050-n)$

$= A(1050-n) + B(n)$

$f(0): 1 = A(1050-0) + B(0)$

$1 = A(1050)$

$\therefore A = \frac{1}{1050}$

$f(1): 1 = A(1050-1) + B(1)$

$1 = \left(\frac{1}{1050}\right)(1049) + B$

$1 - \frac{1049}{1050} = B$

$\therefore B = 9.523809524 \times 10^{-4}$

$\therefore \frac{1/1050}{n} + \frac{1-\frac{1049}{1050}}{1050-n}$

$= \int \left(\frac{1}{1050n} + \frac{1}{1050(1050-n)}\right)dn = \int \frac{1}{5000}\,dt$

$\frac{1}{1050}\int\left(\frac{1}{n} + \frac{1}{1050-n}\right)dn = \int \frac{1}{5000}\,dt$

$\ln n - \ln(1050-n) = \int \frac{1}{5000} \times 1050$

$\ln n - \ln(1050-n) = 0.21t + c$

$\ln\left(\frac{n}{1050-n}\right) = 0.21t + c$ $\left[\ln\left(\frac{n}{1050-n}\right) \ldots\right.$

$\frac{n}{1050-n} = Ae^{0.21t}$ ... $\left.\frac{n}{1050-n} = e^{0.21t} \ldots\right]$

when ... $n = 50$: $\frac{50}{1000} = Ae^{0}$ ...

$\therefore \frac{n}{1050-n} = 0.05e^{0.21t} \Rightarrow n = \frac{1050e^{0.21t}}{20+e^{0.21t}}$

(c) when $n = 250$

$\frac{250}{1050-250} = 0.05e^{0.21t}$

$e^{0.21t} = 6.25$

$0.21t \ln e = \ln 6.25$

$\therefore t = \frac{\ln 6.25}{0.21}$

$= 8.73$ days (3 s.f)

Example 14

The size of a colony of pests, n, which fluctuates during the year, is modelled by an entomologist by the differential equation $\frac{dn}{dt} = 0.2n(0.2 - \cos t)$, where t is the number of weeks from the start of the observations. There are 400 pests in the colony initially.

(a) Solve this differential equation to find n in terms of t.

(b) Find how many pests there are after 3 weeks.

(a) $\frac{dn}{dt} = 0.2n(0.2 - \cos t)$

$\int \frac{5}{n}\,dn = \int (0.2 - \cos t)\,dt$

$5 \ln n = 0.2t - \sin t + c$

$\ln n = 0.2(0.2t - \sin t) + c$ (where $c = \frac{c}{5}$)

$n = Ae^{0.2(0.2t - \sin t)}$ (where $A = e^{c}$)

when $t = 0, n = 400$

$400 = Ae^{0}$

$\therefore A = 400$

$n = 400e^{0.2(0.2t - \sin t)}$

(b) when $t = 3$

$n = 400e^{0.2(0.2(3)) - \sin(3)}$

$n = 438.44$

≈ 438 (3 s.f)