- $\left(4\sec x + 4\sqrt{\cot(\sec y)}\right)dx + \left(3\ln(x) - \sqrt{y\cos y}\right)dy = 0$

$$M_y = 2\,\frac{\sec y\tan y\,(1+\cot^2(\sec y))}{\sqrt{\cot(\sec y)}}$$

$N_x = 3/x$ $\qquad M_y \neq N_x \Rightarrow$ Not exact

Find the Solution by Method Exact:-

$M\,dx + N\,dy = 0 \Rightarrow M_y = N_x$ (Exact)

$\exists\, f(x,y) = c$ S.t $\begin{cases} f_x = M \\ f_y = N \end{cases}$ $\begin{cases} \frac{\partial f}{\partial x} = M \\ \frac{\partial f}{\partial y} = N \end{cases}$

$\frac{\partial f}{\partial x} = M \Rightarrow \partial f = M\,\partial x \Rightarrow f(x,y) = \int M\,dx + g(y) \ldots ①$

$$\begin{cases} \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\int M\,dx\right] + g'(y) \\ \frac{\partial f}{\partial y} = N \end{cases}$$

$\Rightarrow g'(y) = N - \frac{\partial}{\partial y}\left(\int M\,dx\right)$

$\Rightarrow g(y) = \int \left(N - \frac{\partial}{\partial y}\left[\int M\,dx\right]\right)dy \quad \ldots ②$

①, ② $f(x,y) = c$

$My - Nx = 1 + 1 = 2$

$$\mu = e^{\int \frac{My - Nx}{N}} = e^{\int \frac{2}{1-x} dx} = e^{-2\ln|1-x|}$$

$$\mu = \frac{1}{(1-x)^2}$$

$$\mu = \frac{1}{(1-x)^2} \Rightarrow \frac{y}{(1-x)^2} dx + \left(\frac{1}{1-x}\right) dy = 0$$

$$M = \frac{y}{(1-x)^2} \rightarrow My = \frac{1}{(1-x)^2}, \; Nx = \frac{1}{(1-x)^2}$$

it is exact

$\exists f$ such that $f_x = M$ and $f_N = N$

$$\Rightarrow \begin{cases} f_x = \dfrac{y}{(1-x)^2} \\ f_y = \dfrac{1}{1-x} \end{cases} \Rightarrow f(x,y) = \int \frac{y}{1-x} + g(x)$$

$$\Rightarrow \begin{cases} f_x = \dfrac{y}{(1-x)^2} + g'(x) \\ f_N = \dfrac{y}{(1-x)^2} \end{cases}$$

$$\Rightarrow \frac{y}{1-x} = C$$

notice:-

$$d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$$

$$d(xy) = y\,dx + x\,dy$$

$$d(x^2+y^2) = 2x\,dx + 2y\,dy$$

$$d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) = \frac{y\,dx - x\,dy}{x^2+y^2}$$

$$d\left(\ln\left(\frac{x}{y}\right)\right) = \frac{y\,dx - x\,dy}{xy}$$

* $y\,dx - x\,dy = xy^3\,dy$

$$\frac{y\,dx - x\,dy}{xy} = y^2\,dy$$

$$\ln\left(\frac{x}{y}\right) = \frac{1}{3}y^3 + C$$

* $x^2y\,dy - (x\,dy - y\,dx) = 0$

$$x^2y\,d\cancel{x}y = x\,dy - y\,dx$$

$$\frac{-x^2y\,dy}{-x^2} = \frac{y\,dx - x\,dy}{-x^2}$$

$$y\,dy = \frac{x\,dy - y\,dx}{x^2}$$

$$\frac{1}{2}y^2 = y/x + C$$

H.W- $y' = \frac{-x-2y+1}{3(x+2y)}$

H.W- $y' = \frac{x-4y-9}{4x-y+2}$

✓- $y' = \frac{y-2}{x-y-1}$

$X+h=x$

$Y+k=y \Rightarrow Y' = \frac{Y+(k-2)}{X-Y-(h-k-1)}$

$\begin{cases} k-2=0 \\ h-k-1=0 \end{cases} \Rightarrow \begin{matrix} k=2 \\ h=3 \end{matrix} \quad , \quad \begin{matrix} X+3=x \\ Y+2=y \end{matrix} \Rightarrow \begin{cases} X=x-3 \\ Y=y-2 \end{cases}$

$Y' = \frac{Y}{X-Y} = \frac{Y}{X(1-Y/X)} = \frac{Y/X}{1-Y/X}$

$Y/X = Z \Rightarrow Y = XZ$ and $Y' = Z + X\frac{dZ}{dX}$

$Z + X\frac{dZ}{dX} = \frac{Z}{1-Z} \Rightarrow X\frac{dZ}{dX} = \frac{Z^2}{1-Z}$

$\frac{1-Z}{Z^2}dZ = \frac{dX}{X} \Rightarrow (-\frac{1}{Z} - \ln Z) = \ln X + C$

$-\frac{X}{Y} - \ln \frac{Y}{X} - \ln X = C$

$\frac{3-x}{2-y} - \ln\frac{y-2}{x-3} - \ln(x-3) = C \Rightarrow \frac{3-x}{2-y} - \ln(y-2) = C$

How to Solve lagrange equation?

For Solve Lagrange equation, we act like Clairaut equations, we take derivative w.r.t x of the equation $\boxed{y = x\varphi(y') + \psi(y')}$ then
$y' = \varphi(y') + x y'' \varphi'(y') + y'' \psi'(y') = \varphi(y') + y''(x\varphi'(y') + \psi'(y'))$
Now we transform $\varphi(y')$ to another side then
$y' - \varphi(y') = y''(x\varphi'(y') + \psi'(y'))$ (1)

if $y' - \varphi(y') = 0$ then $y'' = 0$ or $x\varphi'(y') + \psi'(y') = 0$

$y'' = 0 \Rightarrow y = c_1 x + c_2$

$x\varphi'(y') + \psi'(y') = 0 \Rightarrow x = \frac{-\psi'(y')}{\varphi'(y')} \Rightarrow y = \frac{-\psi'(y')}{\varphi'(y')}\varphi(y') + \psi(y')$

if $y' - \varphi(y') = 0$ it means that $y' = 0$ or $y' = \varphi(y')$
So $y = c$ or $y' = \varphi(y')$

if $y' - \varphi(y') \neq 0$ Then

$p - \varphi(p) = \frac{dp}{dx} x\varphi'(p) + \frac{dp}{dx}\psi'(p) = \frac{dp}{dx}(x\varphi'(p) + \psi'(p))$

$$\Longrightarrow \frac{dp}{dx} + \frac{-1}{x\varphi'(y') + \psi'(y')} p = \frac{-\varphi(p)}{x\varphi'(y') + \psi'(y')}$$

is a linear equation and the general Solution is $x = f(p, c)$

there fore $\longrightarrow$

$$= \frac{1}{\sqrt{1+x}} \left[\int \frac{1}{2(1+x)} \sqrt{1+x}\, dx + c \right] = 1 + c/\sqrt{1+x}$$

$$y = \int dy = \int (1 + c/\sqrt{1+x})\, dx = (x + 2\sqrt{1+x})$$

H.W * Find the orthogonal trajectory of the family of curves:

$$x^2 - y^2 = 2cx$$

* Solve the following differential equations

1) $y' = x e^{y-x^2}$, $y(0) = 0$

2) $y' = \frac{-x-2y+1}{3(x+2y)}$

3) $y' = x(y')$ [illegible]

4) $y = xy' - 4(y')^3$

Solution 4:- $y = xy' - 4(y')^3$

$$y' = y' + xy'' - 12y''(y')^2$$

$$y''(x - 12(y')^2) = 0 \Rightarrow \begin{cases} y'' = 0 \to y' = c_1 \to y = c_1x + c_2 \\ x = 12(y')^2 \to y' = \sqrt{\frac{x}{12}} \end{cases}$$

$$\begin{cases} y' = c_1 \to y = cx - 4(c)^3 \\ y' = \sqrt{\frac{x}{12}} \to y = \frac{x\sqrt{x}}{2\sqrt{3}} - 4\left(\sqrt{\frac{x}{12}}\right)^3 \end{cases}$$