$$\alpha = 1 - i = \sqrt{2}\big(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\big)$$
$$\beta = 1 + i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$$
$$\alpha^{2n} + \beta^{2n} = \sqrt{2}^{2n}\big(\cos(\frac{-2n\pi}{4}) + i\sin(-\frac{2n\pi}{4}) + \cos(\frac{2n\pi}{4}) + i\sin(\frac{2n\pi}{4})\big)$$
(De Moivre formula)
$$= 2^{n+1}\cos(\frac{2n\pi}{4}) = 2^{n+1}\cos(\frac{n\pi}{2}).$$

6. Suppose non-zero complex numbers z_1, z_2, and z_3 satisfy the relations $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3|$. Prove that

- i)$\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = 0$
- ii)$z_1^2 + z_2^2 + z_3^2 = 0$

Solution.

$$\text{i) } \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = \frac{\bar{z_1}}{z_1\bar{z_1}} + \frac{\bar{z_2}}{z_2\bar{z_2}} + \frac{\bar{z_3}}{z_3\bar{z_3}}$$
$$= \frac{\bar{z_1}}{|z_1|^2} + \frac{\bar{z_2}}{|z_2|^2} + \frac{\bar{z_3}}{|z_3|^2}$$
$$= \frac{1}{|z_1|^2}(\bar{z_1} + \bar{z_2} + \bar{z_3}) = \frac{1}{|z_1|^2}(\overline{z_1 + z_2 + z_3})$$
$$= \frac{1}{|z_1|^2}\cdot 0 = 0.$$

ii) Squaring the equation $z_1 + z_2 + z_3 = 0$, we get that,

$$z_1^2 + z_2^2 + z_3^2 + 2(z_1z_2 + z_2z_3 + z_3z_1) = 0$$
$$z_1^2 + z_2^2 + z_3^2 + 2\times 0 = 0 \qquad \text{(Ref. to i))}$$
$$z_1^2 + z_2^2 + z_3^2 = 0.$$

7.

- i) Prove that $\frac{\sqrt{\tan\theta}+i\sqrt{\cot\theta}}{\sqrt{\tan\theta}-i\sqrt{\cot\theta}} = -(\cos(2\theta) - i\sin(2\theta))$
- ii) Conclude from i) $(\sqrt{\tan\frac{\pi}{20}} + i\sqrt{\cot\frac{\pi}{20}})^{10} + (\sqrt{\tan\frac{\pi}{20}} - i\sqrt{\cot\frac{\pi}{20}})^{10} = 0$

Solution.

$$\text{i) } \frac{\sqrt{\tan\theta}+i\sqrt{\cot\theta}}{\sqrt{\tan\theta}-i\sqrt{\cot\theta}} = \frac{\sqrt{\tan\theta} + \frac{i}{\sqrt{\tan\theta}}}{\sqrt{\tan\theta} - \frac{i}{\sqrt{\tan\theta}}} = \frac{\tan\theta + i}{\tan\theta - i}$$
$$= \frac{\sin\theta + i\cos\theta}{\sin\theta - i\cos\theta} = \frac{i(\cos\theta - i\sin\theta)}{-i(\cos\theta + i\sin\theta)}$$
$$= \frac{ie^{-i\theta}}{-ie^{i\theta}} = -e^{-2i\theta} = -(\cos(2\theta) - i\sin(2\theta)).$$

ii) in i) we put $\theta = \pi/20$, then

$$(\frac{\sqrt{\tan(\pi/20)} + i\sqrt{\cot(\pi/20)}}{\sqrt{\tan(\pi/20)} - i\sqrt{\cot(\pi/20)}})^{10} = \big(-(\cos(\pi/10) - i\sin(\pi/10))\big)^{10}$$