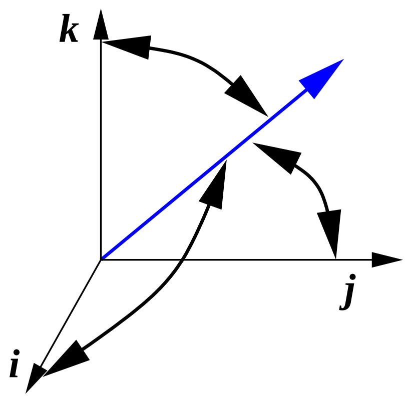

Figure 1.5: The direction cosines are cosines of the angles shown.

It is MUCH more easily remembered in terms of the (pseudo-)determinant

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where the top row consists of the vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$ rather than scalars.

Since a determinant with two equal rows has value zero, it follows that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$. It is also easily verified that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$, so that $\mathbf{a} \times \mathbf{b}$ is orthogonal (perpendicular) to both $\mathbf{a}$ and $\mathbf{b}$ as shown in Figure 1.6.

Note that $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}$, $\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}} = \hat{\boldsymbol{\imath}}$, and $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}}$.

The magnitude of the vector product can be obtained by showing that

$$|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \cdot \mathbf{b})^2 = a^2 b^2$$

from which it follows that

$$|\mathbf{a} \times \mathbf{b}| = ab \sin\theta \ ,$$

which is again independent of the co-ordinate system used. This is left as an exercise.

Unlike the scalar product, the vector product does not satisfy commutativity but is in fact anti-commutative, in that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$. Moreover the vector product does not satisfy the associative law of multiplication either since, as we shall see later $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

Since the vector product is known to be orthogonal to both the vectors which form the product, it merely remains to specify its sense with respect to these vectors. Assuming that the co-ordinate vectors form a right-handed set in the order $\hat{\boldsymbol{\imath}}$,$\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$ it can be seen that the sense of the the vector product is also right handed, i.e

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2.1.4 Vector triple product

This is defined as the vector product of a vector with a vector product, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$. Now, the vector triple product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ must be perpendicular to $(\mathbf{b}\times\mathbf{c})$, which in turn is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$. Thus $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ can have no component perpendicular to $\mathbf{b}$ and $\mathbf{c}$, and hence must be coplanar with them. It follows that the vector triple product must be expressible as a linear combination of $\mathbf{b}$ and $\mathbf{c}$:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \lambda\mathbf{b}+\mu\mathbf{c} \ .$$

The values of the coefficients can be obtained by multiplying out in component form. Only the first component need be evaluated, the others then being obtained by symmetry. That is

$$\begin{aligned}
(\mathbf{a}\times(\mathbf{b}\times\mathbf{c}))_1 &= a_2(\mathbf{b}\times\mathbf{c})_3 - a_3(\mathbf{b}\times\mathbf{c})_2 \\
&= a_2(b_1c_2-b_2c_1)+a_3(b_1c_3-b_3c_1) \\
&= (a_2c_2+a_3c_3)b_1-(a_2b_2+a_3b_3)c_1 \\
&= (a_1c_1+a_2c_2+a_3c_3)b_1-(a_1b_1+a_2b_2+a_3b_3)c_1 \\
&= (\mathbf{a}\cdot\mathbf{c})b_1-(\mathbf{a}\cdot\mathbf{b})c_1
\end{aligned}$$

The equivalents must be true for the 2nd and 3rd components, so we arrive at the identity

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c} \ .$$

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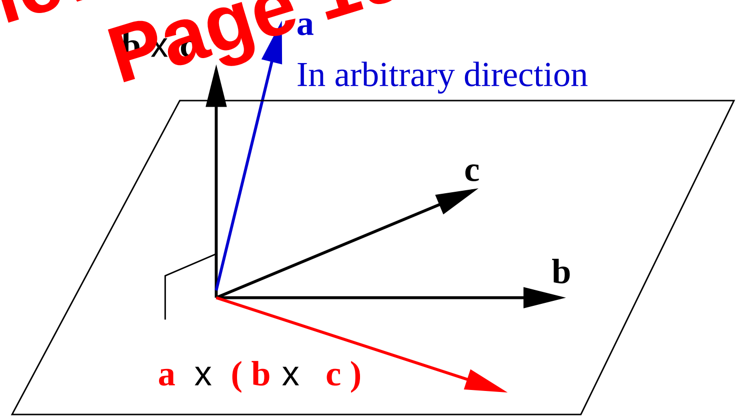

Figure 2.2: Vector triple product.

2.1.5 Projection using vector triple product

An example of the application of this formula is as follows. Suppose $\mathbf{v}$ is a vector and we want its projection into the xy-plane. The component of $\mathbf{v}$ in the z direction is $\mathbf{v}\cdot\hat{\boldsymbol{k}}$, so the projection we seek is $\mathbf{v}-(\mathbf{v}\cdot\hat{\boldsymbol{k}})\hat{\boldsymbol{k}}$. Writing $\hat{\boldsymbol{k}}\leftarrow\mathbf{a}$, $\mathbf{v}\leftarrow\mathbf{b}$,

$\hat{k} \leftarrow \mathbf{c}$,

$$\begin{array}{rcl} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) & = & (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\ \downarrow \quad\quad \downarrow \quad \downarrow & & \\ \hat{k} \times (\mathbf{v} \times \hat{k}) & = & (\hat{k} \cdot \hat{k})\mathbf{v} - (\hat{k} \cdot \mathbf{v})\hat{k} \\ & = & \mathbf{v} - (\mathbf{v} \cdot \hat{k})\hat{k} \end{array}$$

So $\mathbf{v} - (\mathbf{v} \cdot \hat{k})\hat{k} = \hat{k} \times (\mathbf{v} \times \hat{k})$.

(Hot stuff! But the expression $\mathbf{v} - (\mathbf{v} \cdot \hat{k})\hat{k}$ is much easier to understand, and cheaper to compute!)

2.1.6 Other repeated products

Many combinations of vector and scalar products are possible, but we consider only one more, namely the vector quadruple product $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$. By regarding $\mathbf{a} \times \mathbf{b}$ as a single vector, we see that this vector must be representable as a linear combination of $\mathbf{c}$ and $\mathbf{d}$. On the other hand, regarding $\mathbf{c} \times \mathbf{d}$ as a single vector, we see that it must also be a linear combination of $\mathbf{a}$ and $\mathbf{b}$. This provides a means of expressing one of the vectors, say $\mathbf{d}$, as linear combination of the other three, as follows:

$$\begin{array}{rcl} (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) & = & [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\mathbf{c} - [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]\mathbf{d} \\ & = & [(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{a}]\mathbf{b} - [(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{b}]\mathbf{a} \end{array}$$

Hence

$$[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]\,\mathbf{d} = [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d}]\,\mathbf{a} + [(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{d}]\,\mathbf{b} + [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\,\mathbf{c}$$

or

$$\mathbf{d} = \frac{[(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d}]\,\mathbf{a} + [(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{d}]\,\mathbf{b} + [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\,\mathbf{c}}{[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]} \quad = \quad \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} \quad .$$

This is not something to remember off by heart, but it is worth remembering that the projection of a vector on any arbitrary basis set is unique.

♣ Example

Q1 Use the quadruple vector product to express the vector $\mathbf{d} = [3, 2, 1]$ in terms of the vectors $\mathbf{a} = [1, 2, 3]$, $\mathbf{b} = [2, 3, 1]$ and $\mathbf{c} = [3, 1, 2]$.

A1 Grinding away at the determinants, we find

$$[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}] = -18; \quad [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d}] = 6; \quad [(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{d}] = -12; \quad [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}] = -12$$

So, $\mathbf{d} = (-\mathbf{a} + 2\mathbf{b} + 2\mathbf{c})/3$.

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2.2.2 The shortest distance from a point to a line

Referring to Figure 2.5(a) the vector $\mathbf{p}$ from $\mathbf{c}$ to any point on the line is $\mathbf{p} = \mathbf{a} + \lambda\hat{\mathbf{b}} - \mathbf{c} = (\mathbf{a} - \mathbf{c}) + \lambda\hat{\mathbf{b}}$ which has length squared $p^2 = (\mathbf{a} - \mathbf{c})^2 + \lambda^2 + 2\lambda(\mathbf{a} - \mathbf{c}) \cdot \hat{\mathbf{b}}$. Rather than minimizing length, it is easier to minimize length-squared. The minumum is found when $d\ p^2/d\lambda = 0$, ie when

$$\lambda = -(\mathbf{a} - \mathbf{c}) \cdot \hat{\mathbf{b}}\ .$$

So the minimum length vector is

$$\mathbf{p} = (\mathbf{a} - \mathbf{c}) - ((\mathbf{a} - \mathbf{c}) \cdot \hat{\mathbf{b}})\hat{\mathbf{b}}.$$

You might spot that is the component of $(\mathbf{a} - \mathbf{c})$ perpendicular to $\hat{\mathbf{b}}$ (as expected!). Furthermore, using the result of Section 2.1.5,

$$\mathbf{p} = \hat{\mathbf{b}} \times [(\mathbf{a} - \mathbf{c}) \times \hat{\mathbf{b}}]\ .$$

Because $\hat{\mathbf{b}}$ is a unit vector, and is orthogonal to $[(\mathbf{a} - \mathbf{c}) \times \hat{\mathbf{b}}]$, the modulus of the vector can be written rather more simply as just

$$p_{\min} = |(\mathbf{a} - \mathbf{c}) \times \hat{\mathbf{b}}|\ \ .$$

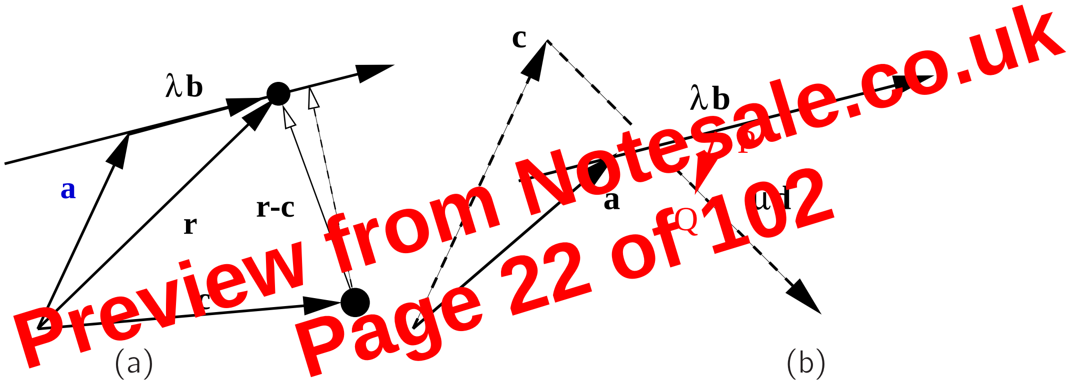

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Figure 2.5: (a) Shortest distance point to line. (b) Shortest distance, line to line.

2.2.3 The shortest distance between two straight lines

If the shortest distance between a point and a line is along the perpendicular, then the shortest distance between the two straight lines $\mathbf{r} = \mathbf{a} + \lambda\hat{\mathbf{b}}$ and $\mathbf{r} = \mathbf{c} + \mu\hat{\mathbf{d}}$ must be found as the length of the vector which is mutually perpendicular to the lines.

The unit vector along the mutual perpendicular is

$$\hat{\mathbf{p}} = (\hat{\mathbf{b}} \times \hat{\mathbf{d}})/|\hat{\mathbf{b}} \times \hat{\mathbf{d}}|\ .$$

(Yes, don't forget that $\hat{\mathbf{b}} \times \hat{\mathbf{d}}$ is NOT a unit vector. $\hat{\mathbf{b}}$ and $\hat{\mathbf{d}}$ are not orthogonal, so there is a $\sin\theta$ lurking!)

The minimum length is therefore the component of $\mathbf{a} - \mathbf{c}$ in this direction

$$p_{\min} = \left|(\mathbf{a} - \mathbf{c}) \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{d}})/|\hat{\mathbf{b}} \times \hat{\mathbf{d}}|\right|\ \ .$$

3.5 Rotating systems

Consider a body which is rotating with constant angular velocity $\boldsymbol{\omega}$ about some axis passing through the origin. Assume the origin is fixed, and that we are sitting in a fixed coordinate system $Oxyz$.

If $\boldsymbol{\rho}$ is a vector of constant magnitude and constant direction in the rotating system, then its representation $\mathbf{r}$ in the fixed system must be a function of t.

$$\mathbf{r}(t) = \mathrm{R}(t)\boldsymbol{\rho}$$

At any instant as observed in the fixed system

$$\frac{d\mathbf{r}}{dt} = \dot{\mathrm{R}}\boldsymbol{\rho} + \mathrm{R}\dot{\boldsymbol{\rho}}$$

but the second term is zero since we assumed $\boldsymbol{\rho}$ to be constant so we have

$$\frac{d\mathbf{r}}{dt} = \dot{\mathrm{R}}\mathrm{R}^{\top}\mathbf{r}$$

Note that:

- $d\mathbf{r}/dt$ will have fixed magnitude;
- $d\mathbf{r}/dt$ will always be perpendicular to the axis of rotation;
- $d\mathbf{r}/dt$ will vary in direction within those constraints;
- $\mathbf{r}(t)$ will move in a plane in the fixed system.

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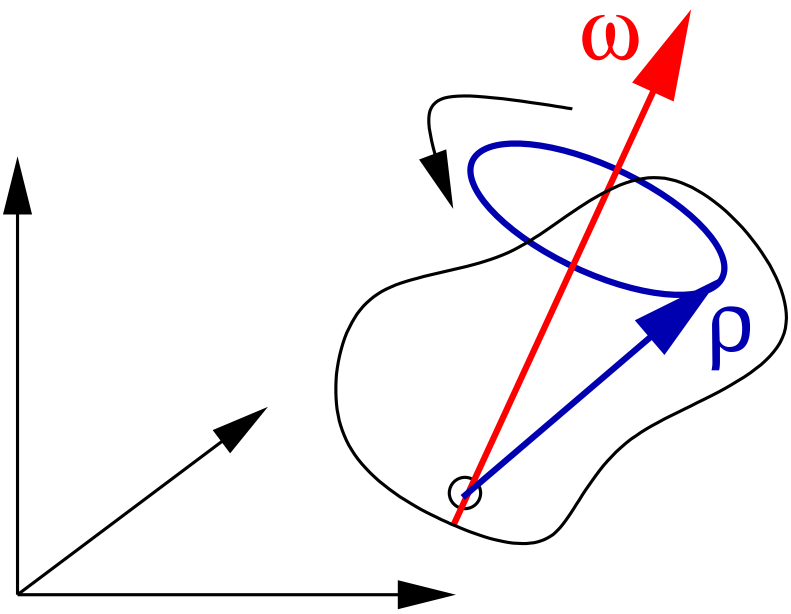

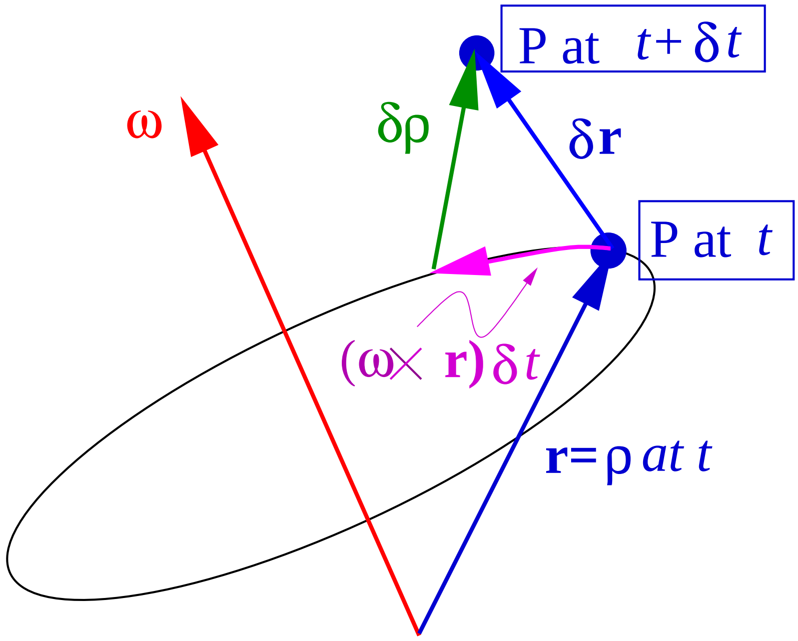

3.5.2 Rotation 3: Instantaneous acceleration

Our previous result is a general one relating the time derivatives of any vector in rotating and non-rotating frames. Let us now consider the second differential:

$$\ddot{\mathbf{r}} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}} + \dot{R}\dot{\boldsymbol{\rho}} + R\ddot{\boldsymbol{\rho}}$$

We shall assume that the angular acceleration is zero, which kills off the first term, and so now, substituting for $\dot{\mathbf{r}}$ we have

$$\begin{aligned}
\ddot{\mathbf{r}} &= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r} + R\dot{\boldsymbol{\rho}}) + \dot{R}\dot{\boldsymbol{\rho}} + R\ddot{\boldsymbol{\rho}} \\
&= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times R\dot{\boldsymbol{\rho}} + \dot{R}\dot{\boldsymbol{\rho}} + R\ddot{\boldsymbol{\rho}} \\
&= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times R\dot{\boldsymbol{\rho}} + \dot{R}(R^{\top}R)\dot{\boldsymbol{\rho}} + R\ddot{\boldsymbol{\rho}} \\
&= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times (R\dot{\boldsymbol{\rho}}) + R\ddot{\boldsymbol{\rho}}
\end{aligned}$$

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The **instantaneous acceleration** is therefore

$$\ddot{\mathbf{r}} = R\ddot{\boldsymbol{\rho}} + 2\boldsymbol{\omega} \times (R\dot{\boldsymbol{\rho}}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

- The first term is the acceleration of the point P in the rotating frame measured in the rotating frame, but referred to the fixed frame by the rotation R

- The last term is the centripetal acceleration to due to the rotation. (Yes! Its magnitude is $\omega^2 r$ and its direction is that of $-\mathbf{r}$. Check it out.)

last term is the centripetal acceleration resulting from the rotation of the sphere. The middle term is the Coriolis acceleration.

Using Fig. 3.2, at some instant t

$$\mathbf{r}(t) = \boldsymbol{\rho}(t) = r\cos(\gamma t)\hat{\mathbf{m}} \quad + \quad r\sin(\gamma t)\hat{\mathbf{n}}$$

and

$$\boldsymbol{\gamma} = \gamma\hat{\boldsymbol{\ell}}$$

Then

$$\boldsymbol{\gamma} \times (\boldsymbol{\gamma} \times \boldsymbol{\rho}) = (\boldsymbol{\gamma} \cdot \boldsymbol{\rho})\boldsymbol{\gamma} - \gamma^2\boldsymbol{\rho} = -\gamma^2\boldsymbol{\rho} = -\gamma^2\mathbf{r},$$

Check the direction — the negative sign means it points *towards* the centre of the sphere, which is as expected.

Likewise the last term can be obtained as

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = -\omega^2 r\sin(\gamma t)\hat{\mathbf{n}}$$

Note that it is perpendicular to the axis of rotation $\hat{\mathbf{m}}$, and because of the minus sign, directed towards the axis)

The Coriolis term is derived as:

$$\begin{aligned} 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}} &= 2\boldsymbol{\omega} \times (\boldsymbol{\gamma} \times \boldsymbol{\rho}) \\ &= 2\begin{bmatrix} 0 \\ \omega \\ 0 \end{bmatrix} \times \left(\begin{bmatrix} \gamma \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ r\cos\gamma t \\ r\sin\gamma t \end{bmatrix} \right) \\ &= 2\omega\gamma r\cos\gamma t\hat{\boldsymbol{\ell}} \end{aligned}$$

Instead of a projectile, now consider a rocket on rails which stretch north from the equator. As the rocket travels north it experiences the Coriolis force (exerted by the rails):

$$\begin{array}{ccccc} 2 & \gamma & \omega & R\cos\gamma t & \hat{\boldsymbol{\ell}} \\ +\text{ve} & \text{-ve} & +\text{ve} & +\text{ve} & \end{array}$$

Hence the coriolis force is in the direction opposed to $\hat{\boldsymbol{\ell}}$ (i.e. in the opposite direction to the earth's rotation). In the absence of the rails (or atmosphere) the rocket's tangetial speed (relative to the surface of the earth) is *greater* than the speed of the surface of the earth underneath it (since the radius of successive lines of latitude decreases) so it would (to an observer on the earth) appear to deflect to the east. The rails provide a coriolis force keeping it on the same meridian.

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at point $\mathbf{r}$ and the instantaneous displacement along curve C is $d\mathbf{r}$ then the infinitessimal work done is $dW = \mathbf{F}.d\mathbf{r}$, and so the total work done traversing the path is

$$W_C = \int_C \mathbf{F}.d\mathbf{r}$$

- Ampère's law relating magnetic field $\mathbf{B}$ to linked current can be written as

  $$\oint_C \mathbf{B}.d\mathbf{r} = \mu_0 I$$

  where I is the current enclosed by (closed) path C.

- The force on an element of wire carrying current I, placed in a magnetic field of strength $\mathbf{B}$, is $d\mathbf{F} = Id\mathbf{r} \times \mathbf{B}$. So if a loop this wire C is placed in the field then the total force will be and integral of type 3 above:

  $$\mathbf{F} = I \oint_C d\mathbf{r} \times \mathbf{B}$$

Note that the expressions above are beautifully compact in vector notation, and are all independent of coordinate system. Of course when evaluating them we need to choose a coordinate system: often this is the standard Cartesian coordinate system (as in the worked examples below), but need not be, as we shall see in section 4.6.

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♣ Examples

Q1 An example in the xy-plane. A force $\mathbf{F} = x^2y\hat{\imath} + xy^2\hat{\jmath}$ acts on a body as it moves between $(0,0)$ and $(1,1)$.

Determine the work done when the path is

1. along the line $y = x$.
2. along the curve $y = x^n$, $n > 0$.
3. along the x axis to the point $(1,0)$ and then along the line $x = 1$.

A1 This is an example of the "type 2" line integral. In planar Cartesians, $d\mathbf{r} = \hat{\imath}dx + \hat{\jmath}dy$. Then the work done is

$$\int_L \mathbf{F}\cdot d\mathbf{r} = \int_L (x^2y dx + xy^2 dy) \ .$$

1. For the path $y = x$ we find that $dy = dx$. So it is easiest to convert all y references to x.

$$\int_{(0,0)}^{(1,1)} (x^2y dx + xy^2 dy) = \int_{x=0}^{x=1} (x^2x dx + xx^2 dx) = \int_{x=0}^{x=1} 2x^3 dx = \left[x^4/2\right|_{x=0}^{x=1} = 1/2 \, .$$

in (eg) x while keeping y and z constant would result in a displacement of

$$ds = |d\mathbf{r}| = \sqrt{d\mathbf{r}.d\mathbf{r}} = \sqrt{dx^2 + 0 + 0} = dx$$

But in cylindrical polars, a small change in ϕ of $d\phi$ while keeping r and z constant results in a displacement of

$$ds = |d\mathbf{r}| = \sqrt{r^2(d\phi)^2} = r d\phi$$

Thus the size of the (infinitessimal) displacement is dependent on the value of r. Factors such as this r are known as **scale factors** or **metric coefficients**, and we must be careful to take them into account when, eg, performing line, surface or volume integrals, as you will below. For cylindrical polars the metric coefficients are clearly 1, r and 1.

Example: line integral in cylindrical coordinates

Q Evaluate $\oint_C \mathbf{a} \cdot d\mathbf{l}$, where $\mathbf{a} = x^3\hat{\boldsymbol{j}} - y^3\hat{\boldsymbol{\imath}} + x^2y\hat{\boldsymbol{k}}$ and C is the circle of radius r in the $z = 0$ plane, centred on the origin.

A Consider figure 4.5. In this case our cylindrical coordinates effectively reduce to plane polars since the path of integration is a circle in the $z = 0$ plane, but let's persist with the full set of coordinates anyway; the $\hat{\boldsymbol{k}}$ component of $\mathbf{a}$ will play no role (it is normal to the path of integration and therefore cancels as seen below).

On the circle of interest

$$\mathbf{a} = r^3(-\sin^3\phi\hat{\boldsymbol{\imath}} + \cos^3\phi\hat{\boldsymbol{j}} + \cos^2\phi\sin\phi\hat{\boldsymbol{k}})$$

and (since dz = dr = 0 on the path)

$$\begin{aligned} d\mathbf{r} &= r\ d\phi\ \hat{\mathbf{e}}_\phi \\ &= r d\phi(-\sin\phi\hat{\boldsymbol{\imath}} + \cos\phi\hat{\boldsymbol{j}}) \end{aligned}$$

so that

$$\oint_C \mathbf{a} \cdot d\mathbf{r} = \int_0^{2\pi} r^4(\sin^4\phi + \cos^4\phi) d\phi = \frac{3\pi}{2} r^4$$

since

$$\int_0^{2\pi} \sin^4\phi d\phi = \int_0^{2\pi} \cos^4\phi d\phi = \frac{3\pi}{4}$$

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The metric coefficients are therefore $h_u = |\frac{\partial \mathbf{r}}{\partial u}|$, $h_v = |\frac{\partial \mathbf{r}}{\partial v}|$ and $h_w = |\frac{\partial \mathbf{r}}{\partial w}|$.
A volume element is in general given by

$$dV = h_u du \hat{\mathbf{e}}_u.(h_v dv \hat{\mathbf{e}}_v \times h_w dw \hat{\mathbf{e}}_w)$$

and simplifies *if the coordinate system is orthonormal* (since $\hat{\mathbf{e}}_u.(\hat{\mathbf{e}}_v \times \hat{\mathbf{e}}_w) = 1$) to

$$dV = h_u h_v h_w du dv dw$$

A surface element (normal to constant w, say) is in general

$$d\mathbf{S} = h_u du \hat{\mathbf{e}}_u \times h_v dv \hat{\mathbf{e}}_v$$

and simplifies *if the coordinate system is orthogonal* to

$$d\mathbf{S} = h_u h_v du dv \hat{\mathbf{e}}_w$$

4.6.4 Summary

To summarise:

General curvilinear coordinates

$$\begin{aligned}
x &= x(u,v,w), \qquad y = y(u,v,w), \qquad z = z(u,v,w) \\
\mathbf{r} &= x(u,v,w)\hat{\imath} + y(u,v,w)\hat{\jmath} + z(u,v,w)\hat{k} \\
h_u &= \left|\frac{\partial \mathbf{r}}{\partial u}\right|, \qquad h_v = \left|\frac{\partial \mathbf{r}}{\partial v}\right|, \qquad h_w = \left|\frac{\partial \mathbf{r}}{\partial w}\right| \\
\hat{\mathbf{u}} &= \hat{\mathbf{e}}_u = \frac{1}{h_u}\frac{\partial \mathbf{r}}{\partial u}, \qquad \hat{\mathbf{v}} = \hat{\mathbf{e}}_v = \frac{1}{h_v}\frac{\partial \mathbf{r}}{\partial v}, \qquad \hat{\mathbf{w}} = \hat{\mathbf{e}}_w = \frac{1}{h_w}\frac{\partial \mathbf{r}}{\partial w} \\
d\mathbf{r} &= h_u du \hat{\mathbf{u}} + h_v dv \hat{\mathbf{v}} + h_w dw \hat{\mathbf{w}} \\
dV &= h_u h_v h_w du dv dw \ \hat{\mathbf{u}}.(\hat{\mathbf{v}} \times \hat{\mathbf{w}}) \\
d\mathbf{S} &= h_u h_v du dv \ \hat{\mathbf{u}} \times \hat{\mathbf{v}} \quad \text{(for surface element tangent to constant } w\text{)}
\end{aligned}$$

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Plane polar coordinates

$$\begin{aligned}
x &= r\cos\theta, \qquad y = r\sin\theta \\
\mathbf{r} &= r\cos\theta\hat{\imath} + r\sin\theta\hat{\jmath} \\
h_r &= 1, \qquad h_\theta = r \\
\hat{\mathbf{e}}_r &= \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}, \qquad \hat{\mathbf{e}}_\theta = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath} \\
d\mathbf{r} &= dr\hat{\mathbf{e}}_r + rd\theta\hat{\mathbf{e}}_\theta \\
d\mathbf{S} &= rdrd\theta\hat{\mathbf{k}}
\end{aligned}$$

Lecture 5

Vector Operators: Grad, Div and Curl

In the first lecture of the second part of this course we move more to consider properties of fields. We introduce three field operators which reveal interesting collective field properties, viz.

- the **gradient** of a scalar field,
- the **divergence** of a vector field, and
- the **curl** of a vector field.

There are two points to get over about each:

- The mechanics of taking the grad, div or curl, for which you will need to brush up your multivariate calculus.
- The underlying physical meaning — that is, why they are worth bothering about.

In Lecture 6 we will look at combining these vector operators.

5.1 The gradient of a scalar field

Recall the discussion of temperature distribution throughout a room in the overview, where we wondered how a scalar would vary as we moved off in an arbitrary direction. Here we find out how.

If $U(x, y, z)$ is a scalar field, ie a scalar function of position $\mathbf{r} = [x, y, z]$ in 3 dimensions, then its **gradient** at any point is defined in Cartesian co-ordinates by

$$\mathrm{grad}U = \frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{\boldsymbol{k}} .$$

It is usual to define the **vector operator** which is called "del" or "nabla"

$$\boldsymbol{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z} .$$

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Then

$$\text{grad}U \equiv \nabla U \ .$$

Note immediately that ∇U is a vector field!

Without thinking too carefully about it, we can see that the gradient of a scalar field tends to point in the direction of greatest change of the field. Later we will be more precise.

♣ Worked examples of gradient evaluation

1. $U = x^2$

$$\Rightarrow \nabla U = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}} \right) x^2 = 2x\hat{\imath} \ .$$

2. $U = r^2$

$$\begin{aligned} r^2 &= x^2 + y^2 + z^2 \\ \Rightarrow \nabla U &= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}} \right)(x^2 + y^2 + z^2) \\ &= 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{\boldsymbol{k}} = 2\mathbf{r} \ . \end{aligned}$$

3. $U = \mathbf{c} \cdot \mathbf{r}$, where $\mathbf{c}$ is constant.

$$\Rightarrow \nabla U = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z} \right)(c_1 x + c_2 y + c_3 z) = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{\boldsymbol{k}} = \mathbf{c} \ .$$

4. $U = f(r)$, where $r = \sqrt{(x^2 + y^2 + z^2)}$

U is a function of r alone so df/dr exists. As $U = f(x, y, z)$ also,

$$\frac{\partial f}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x} \qquad \frac{\partial f}{\partial y} = \frac{df}{dr}\frac{\partial r}{\partial y} \qquad \frac{\partial f}{\partial z} = \frac{df}{dr}\frac{\partial r}{\partial z} \ .$$

$$\Rightarrow \nabla U = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}} \quad = \frac{df}{dr}\left(\frac{\partial r}{\partial x}\hat{\imath} + \frac{\partial r}{\partial y}\hat{\jmath} + \frac{\partial r}{\partial z}\hat{\boldsymbol{k}} \right)$$

But $r = \sqrt{x^2 + y^2 + z^2}$, so $\partial r/\partial x = x/r$ and similarly for y, z.

$$\Rightarrow \nabla U = \frac{df}{dr}\left(\frac{x\hat{\imath} + y\hat{\jmath} + z\hat{\boldsymbol{k}}}{r} \right) = \frac{df}{dr}\left(\frac{\mathbf{r}}{r} \right) \ .$$

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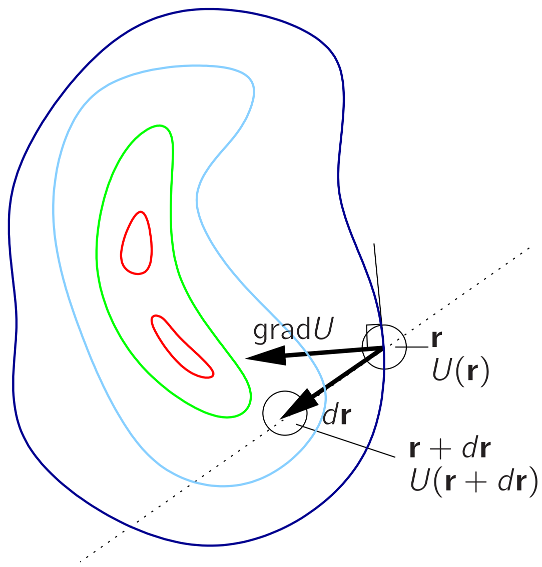

Figure 5.1: The directional derivative

5.2 The significance of grad

If our current position is $\mathbf{r}$ in some scalar field U (Fig. 5.1), and we move an infinitesimal distance $d\mathbf{r}$, we know that the change in U is

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz \ .$$

But we know that $d\mathbf{r} = (\hat{\imath}dx + \hat{\jmath}dy + \hat{\mathbf{k}}dz)$ and $\nabla U = (\hat{\imath}\partial U/\partial x + \hat{\jmath}\partial U/\partial y + \hat{\mathbf{k}}\partial U/\partial z)$, so that the change in U is also given by the scalar product

$$dU = \nabla U \cdot d\mathbf{r} \ .$$

Now divide both sides by ds

$$\frac{dU}{ds} = \nabla U \cdot \frac{d\mathbf{r}}{ds} \ .$$

But remember that $|d\mathbf{r}| = ds$, so $d\mathbf{r}/ds$ is a unit vector in the direction of $d\mathbf{r}$.

This result can be paraphrased as:

- grad$U$ has the property that the rate of change of U wrt distance in a particular direction ($\hat{\mathbf{d}}$) is the projection of grad$U$ onto that direction (or the component of grad$U$ in that direction).

The quantity dU/ds is called a **directional derivative**, but note that in general it has a different value for each direction, and so has no meaning until you specify the direction.

We could also say that

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7.7 An Extension to Stokes' Theorem

Just as we considered one extension to Gauss' theorem (not *really* an extension, more of a re-expression), so we will try something similar with Stoke's Theorem.

Again let $\mathbf{a}(\mathbf{r}) = U(\mathbf{r})\mathbf{c}$, where $\mathbf{c}$ is a constant vector. Then

$$\text{curl } \mathbf{a} = U \text{curl } \mathbf{c} + \text{grad } U \times \mathbf{c})$$

Again, curl $\mathbf{c}$ is zero. Stokes' Theorem becomes in this case:

$$\oint_C U(\mathbf{c} \cdot d\mathbf{l}) = \int_S (\text{grad } U \times \mathbf{c} \cdot d\mathbf{S} = \int_S \mathbf{c} \cdot (d\mathbf{S} \times \text{grad } U)$$

or, rearranging the triple scalar products and taking the constant $\mathbf{c}$ out of the integrals gives

$$\mathbf{c} \cdot \oint_C U d\mathbf{l} = -\mathbf{c} \cdot \int_S \text{grad} U \times d\mathbf{S} \quad .$$

But $\mathbf{c}$ is arbitrary and so

$$\oint_C U d\mathbf{l} = -\int_S \text{grad } U \times d\mathbf{S}$$

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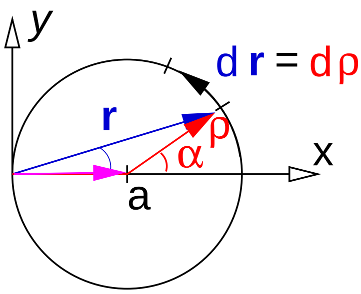
7.8 ♣ Example of extension to Stokes' Theorem

Q Derive $\oint_C U d\mathbf{r}$ (i) directly and (ii) using Stokes', where $U = x^2 + y^2 + z^2$ and the line integral is taken around C the circle $(x-a)^2 + y^2 = a^2$ and $z = 0$.

Note that, for no special reason, we have used $d\mathbf{r}$ here not $d\mathbf{l}$.



A(i) First some preamble.

If the circle were centred at the origin, we would write $d\mathbf{r} = a d\theta \hat{\mathbf{e}}_\theta = a d\theta(-\sin\theta \hat{\imath} + \cos\theta \hat{\jmath})$. For such a circle the magnitude $r = |\mathbf{r}| = a$, a constant and so $dr = 0$.

However, in this example $d\mathbf{r}$ is not always in the direction of $\hat{\mathbf{e}}_\theta$, and $dr \neq 0$. Could you write down $d\mathbf{r}$? If not, revise Lecture 3, where we saw that in plane polars $x = r\cos\theta$, $y = r\sin\theta$ and the general expression is

$$d\mathbf{r} = dx\hat{\imath} + dy\hat{\jmath} = (\cos\theta dr - r\sin\theta d\theta)\hat{\imath} + (\sin\theta dr + r\cos\theta d\theta)\hat{\jmath}$$

Lecture 8

Engineering applications

In Lecture 6 we saw one classic example of the application of vector calculus to Maxwell's equation.

In this lecture we explore a few more examples from fluid mechanics and heat transfer. As with Maxwell's eqations, the examples show how vector calculus provides a powerful way of representing underlying physics.

The power come from the fact that div, grad and curl have a significance or meaning which is more immediate than a collection of partial derivatives. Vector calculus will, with practice, become a convenient shorthand for you.

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- Electricity – Ampère's Law
- Fluid Mechanics - The Continuity Equation
- Thermo: The Heat Conduction Equation
- Mechanics/Electrostatics - Conservative fields
- The Inverse Square Law of force
- Gravitational field due to distributed mass
- Gravitational field inside body
- Pressure forces in non-uniform flows

8.2 Fluid Mechanics - The Continuity Equation

The **Continuity Equation** expresses the condition of conservation of mass in a fluid flow. The continuity principle applied to *any* volume (called a *control volume*) may be expressed in words as follows:

> "The net rate of mass flow of fluid out of the control volume must equal the rate of decrease of the mass of fluid within the control volume"

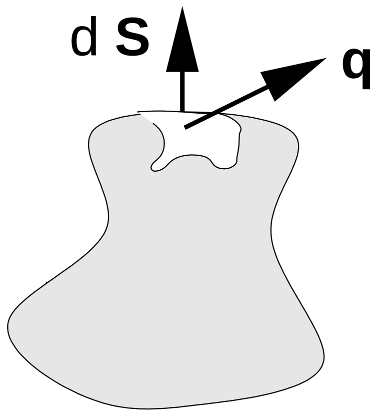

Figure 8.1:

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To express the above as a mathematical equation we denote the velocity of the fluid at each point of the flow by $\mathbf{q}(\mathbf{r})$ (a vector field) and the density by $\rho(\mathbf{r})$ (a scalar field). The element of rate-of-volume-loss through surface $d\mathbf{S}$ is $d\dot{V} = \mathbf{q}\cdot d\mathbf{S}$, so the rate of mass loss is

$$d\dot{M} = \rho\mathbf{q}\cdot d\mathbf{S},$$

so that the total rate of mass loss from the volume is

$$-\frac{\partial}{\partial t}\int_V \rho(\mathbf{r})dV = \int_S \rho\mathbf{q}\cdot d\mathbf{S}.$$

Assuming that the volume of interest is fixed, this is the same as

$$-\int_V \frac{\partial\rho}{\partial t}dV = \int_S \rho\mathbf{q}\cdot d\mathbf{S}\ .$$

Now we use Gauss' Theorem to transform the RHS into a volume integral

$$-\int_V \frac{\partial\rho}{\partial t}dV = \int_V \mathrm{div}\ (\rho\mathbf{q})dV\ .$$

The two volume integrals can be equal for any control volume V only if the two integrands are equal at each point of the flow. This leads to the mathematical formulation of

8.5 The Inverse Square Law of force

Radial forces are found in electrostatics and gravitation — so they are certainly irrotational and conservative.

But in nature these radial forces are also inverse square laws. One reason why this may be so is that it turns out to be the only central force field which is **solenoidal**, i.e. has zero divergence.

If $\mathbf{F} = f(r)\mathbf{r}$,

$$\text{div } \mathbf{F} = 3f(r) + rf'(r).$$

For div $\mathbf{F} = 0$ we conclude

$$r\frac{df}{dr} + 3f = 0$$

or

$$\frac{df}{f} + 3\frac{dr}{r} = 0.$$

Integrating with respect to r gives $fr^3 = const = A$, so that

$$\mathbf{F} = \frac{A\mathbf{r}}{r^3}, \quad |\mathbf{F}| = \frac{A}{r^2}.$$

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The condition of zero divergence of the inverse square force field applies everywhere except at $\mathbf{r} = \mathbf{0}$, where the divergence is infinite.

To show this, calculate the outward normal flux out of a sphere of radius R centered on the origin when $\mathbf{F} = F\hat{\mathbf{r}}$. This is

$$\int_S \mathbf{F} \cdot d\mathbf{S} = F\int_{\text{Sphere}} \hat{\mathbf{r}} \cdot d\mathbf{S} = F\int_{\text{Sphere}} d = F4\pi R^2 = 4\pi A = \text{Constant}.$$

Gauss tells us that this flux must be equal to

$$\int_V \text{div } \mathbf{F} dV = \int_0^R \text{div } \mathbf{F} 4\pi r^2 dr$$

where we have done the volume integral as a summation over thin shells of surface area $4\pi r^2$ and thickness dr.

But for all finite r, div$\mathbf{F} = 0$, so div$\mathbf{F}$ must be infinite at the origin.

The flux integral is thus

- zero — for any volume which does not contain the origin
- $4\pi A$ for any volume which does contain it.