A Term of
Commutative Algebra

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Given $T \subset X$, clearly $\chi_S \cdot \chi_T = \chi_{S\cap T}$. Further, $\chi_S + \chi_T = \chi_{S\triangle T}$, where $S\triangle T$ is the **symmetric difference**:

$$S\triangle T := (S \cup T) - (S \cap T) = (S - T) \cup (T - S);$$

here $S - T$ denotes, as usual, the set of elements of S not in T. Thus *the subsets of X form a ring: sum is symmetric difference, and product is intersection. This ring is canonically isomorphic to $\mathbb{F}_2^X$.*

A ring B is said to be **Boolean** if $f^2 = f$ for all $f \in B$. Clearly, $\mathbb{F}_2^X$ is Boolean.

Suppose X is a topological space, and give $\mathbb{F}_2$ the **discrete** topology; that is, every subset is both open and closed. Consider the continuous functions $f\colon X \to \mathbb{F}_2$. Clearly, they are just the χ_S where S is both open and closed. Clearly, they form a Boolean subring of $\mathbb{F}_2^X$. Conversely, Stone's Theorem (**13.25**) asserts that *every Boolean ring is canonically isomorphic to the ring of continuous functions from a compact Hausdorff topological space X to $\mathbb{F}_2$, or equivalently, isomorphic to the ring of open and closed subsets of X.*

(1.3) (*Polynomial rings*). — Let R be a ring, $P := R[X_1, \dots, X_n]$ the polynomial ring in n variables (see [2, pp. 352–3] or [8, p. 268]). Recall that P has this **Universal Mapping Property** (UMP): *given a ring map $\varphi\colon R \to R'$ and given an element x_i of R' for each i, there is a unique ring map $\pi\colon P \to R'$ with $\pi|R = \varphi$ and $\pi(X_i) = x_i$.* In fact, since π is a ring map, necessarily π is given by the formula:

$$\pi\Big(\sum a_{(i_1,\dots,i_n)} X_1^{i_1} \cdots X_n^{i_n}\Big) = \sum \varphi(a_{(i_1,\dots,i_n)}) x_1^{i_1} \cdots x_n^{i_n}.$$

In other words, P is universal among R-algebras equipped with a list of n elements: P is one, and it maps uniquely to any other.

Similarly, let $P' := R[\{X_\lambda\}_{\lambda\in\Lambda}]$ be the polynomial ring in an arbitrary list of variables: its elements are the polynomials in any finitely many of the X_λ; sum and product are defined as in P. Thus P' contains as a subring the polynomial ring in any finitely many X_λ, and P' is the union of these subrings. Clearly, P' has essentially the same UMP as P: *given $\varphi\colon R \to R'$ and given $x_\lambda \in R'$ for each λ, there is a unique $\pi\colon P' \to R'$ with $\pi|R = \varphi$ and $\pi(X_\lambda) = x_\lambda$.*

(1.4) (*Ideals*). — Let R be a ring. Recall that a subset $\mathfrak{a}$ is called an **ideal** if

(1) $0 \in \mathfrak{a}$,
(2) whenever $a, b \in \mathfrak{a}$, also $a + b \in \mathfrak{a}$, and
(3) whenever $x \in R$ and $a \in \mathfrak{a}$, also $xa \in \mathfrak{a}$.

Given elements $a_\lambda \in R$ for $\lambda \in \Lambda$, by the ideal $\langle a_\lambda\rangle_{\lambda\in\Lambda}$ they **generate**, we mean the smallest ideal containing them all. If $\Lambda = \emptyset$, then this ideal consists just of 0.

Any ideal containing all the a_λ contains any (finite) **linear combination** $\sum x_\lambda a_\lambda$ with $x_\lambda \in R$ and almost all 0. Form the set $\mathfrak{a}$, or $\sum R a_\lambda$, of all such linear combinations; clearly, $\mathfrak{a}$ is an ideal containing all a_λ. Thus $\mathfrak{a}$ is the ideal generated by the a_λ.

Given a single element a, we say that the ideal $\langle a\rangle$ is **principal**. By the preceding observation, $\langle a\rangle$ is equal to the set of all multiples xa with $x \in R$.

Similarly, given ideals $\mathfrak{a}_\lambda$ of R, by the ideal they generate, we mean the smallest ideal $\sum \mathfrak{a}_\lambda$ that contains them all. Clearly, $\sum \mathfrak{a}_\lambda$ is equal to the set of all finite linear combinations $\sum x_\lambda a_\lambda$ with $x_\lambda \in R$ and $a_\lambda \in \mathfrak{a}_\lambda$.

$\varphi(\mathfrak{a}) = 0$, *there is a unique ring map* $\psi\colon R/\mathfrak{a} \to R'$ *such that* $\psi\kappa = \varphi$. In other words, $R/\mathfrak{a}$ is universal among R-algebras R' such that $\mathfrak{a}R' = 0$.

Above, if $\mathfrak{a}$ is the ideal generated by elements a_λ, then the UMP can be usefully rephrased as follows: $\kappa(a_\lambda) = 0$ *for all* λ, *and given* $\varphi\colon R \to R'$ *such that* $\varphi(a_\lambda) = 0$ *for all* λ, *there is a unique ring map* $\psi\colon R/\mathfrak{a} \to R'$ *such that* $\psi\kappa = \varphi$.

The UMP serves to determine $R/\mathfrak{a}$ *up to unique isomorphism.* Indeed, say R', equipped with $\varphi\colon R \to R'$, has the UMP too. Then $\varphi(\mathfrak{a}) = 0$; so there is a unique $\psi\colon R/\mathfrak{a} \to R'$ with $\psi\kappa = \varphi$. And $\kappa(\mathfrak{a}) = 0$; so there is a unique $\psi'\colon R' \to R/\mathfrak{a}$ with $\psi'\varphi = \kappa$. Then, as shown, $(\psi'\psi)\kappa = \kappa$, but $1 \circ \kappa = \kappa$ where 1

$$\begin{tikzcd} & & R/\mathfrak{a} \arrow[dl, "\psi"] \arrow[dd, "1"] \\ R \arrow[urr, "\kappa"] \arrow[r, "\varphi"] \arrow[drr, "\kappa"'] & R' \arrow[dr, "\psi'"] & \\ & & R/\mathfrak{a} \end{tikzcd}$$

is the identity map of $R/\mathfrak{a}$; hence, $\psi'\psi = 1$ by uniqueness. Similarly, $\psi\psi' = 1$ where 1 now stands for the identity map of R'. Thus ψ and ψ' are inverse isomorphisms.

The preceding proof is completely formal, and so works widely. There are many more constructions to come, and each one has an associated UMP, which therefore serves to determine the construction up to unique isomorphism.

Exercise (**1.7**). — Let R be a ring, $\mathfrak{a}$ an ideal, and $P := R[X_1, \ldots, X_n]$ the polynomial ring. Prove $P/\mathfrak{a}P = (R/\mathfrak{a})[X_1, \ldots, X_n]$.

Proposition (**1.8**). — *Let* R *be a ring,* $P := R[X]$ *the polynomial ring in one variable,* $a \in R$, *and* $\pi\colon P \to R$ *the* R*-algebra map defined by* $\pi(X) := a$. *Then* $\operatorname{Ker}(\pi) = \langle X - a\rangle$, *and* $R[X]/\langle X - a\rangle \xrightarrow{\sim} R$.

Proof: Given $F(X) \in P$, the Division Algorithm yields $F(X) = G(X)(X-a)+b$ with $G(X) \in P$ and $b \in R$. Then $\pi(F(X)) = b$. Hence $\operatorname{Ker}(\pi) = \langle X - a\rangle$. Finally, (**1.6.1**) yields $R[X]/\langle X - a\rangle \xrightarrow{\sim} R$. □

(1.9) (*Nested ideals*). — Let R be a ring, $\mathfrak{a}$ an ideal, and $\kappa\colon R \to R/\mathfrak{a}$ the quotient map. Given an ideal $\mathfrak{b} \supset \mathfrak{a}$, form the corresponding set of cosets of $\mathfrak{a}$:

$$\mathfrak{b}/\mathfrak{a} := \{b + \mathfrak{a} \mid b \in \mathfrak{b}\} = \kappa(\mathfrak{b}).$$

Clearly, $\mathfrak{b}/\mathfrak{a}$ is an ideal of $R/\mathfrak{a}$. Also $\mathfrak{b}/\mathfrak{a} = \mathfrak{b}(R/\mathfrak{a})$.

Clearly, *the operations* $\mathfrak{b} \mapsto \mathfrak{b}/\mathfrak{a}$ *and* $\mathfrak{b}' \mapsto \kappa^{-1}(\mathfrak{b}')$ *are inverse to each other, and establish a bijective correspondence between the set of ideals* $\mathfrak{b}$ *of* R *containing* $\mathfrak{a}$ *and the set of all ideals* $\mathfrak{b}'$ *of* $R/\mathfrak{a}$. *Moreover, this correspondence preserves inclusions.*

Given an ideal $\mathfrak{b} \supset \mathfrak{a}$, form the composition of the quotient maps

$$\varphi\colon R \to R/\mathfrak{a} \to (R/\mathfrak{a})\big/(\mathfrak{b}/\mathfrak{a}).$$

Clearly, φ is surjective, and $\operatorname{Ker}(\varphi) = \mathfrak{b}$. Hence, owing to (**1.6**), φ factors through the canonical isomorphism ψ in this commutative diagram:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R/\mathfrak{b} \arrow[d, "\psi"', "\simeq"] \\ R/\mathfrak{a} \arrow[r] & (R/\mathfrak{a})\big/(\mathfrak{b}/\mathfrak{a}) \end{tikzcd}$$

Exercise (1.10). — Let R be ring, and $P := R[X_1, \dots, X_n]$ the polynomial ring. Let $m \le n$ and $a_1, \dots, a_m \in R$. Set $\mathfrak{p} := \langle X_1 - a_1, \dots, X_m - a_m \rangle$. Prove that $P/\mathfrak{p} = R[X_{m+1}, \dots, X_n]$.

(1.11) (*Idempotents*). — Let R be a ring. Let $e \in R$ be an **idempotent**; that is, $e^2 = e$. Then Re is a ring with e as 1, because $(xe)e = xe$. But Re is not a subring of R unless $e = 1$, although Re is an ideal.

Set $e' := 1 - e$. Then e' is idempotent and $e \cdot e' = 0$. We call e and e' **complementary idempotents**. Conversely, if two elements $e_1, e_2 \in R$ satisfy $e_1 + e_2 = 1$ and $e_1 e_2 = 0$, then they are complementary idempotents, as for each i,

$$e_i = e_i \cdot 1 = e_i(e_1 + e_2) = e_i^2.$$

We denote the set of all idempotents by $\mathrm{Idem}(R)$. Let $\varphi\colon R \to R'$ be a ring map. Then $\varphi(e)$ is idempotent. So the restriction of φ to $\mathrm{Idem}(R)$ is a map

$$\mathrm{Idem}(\varphi)\colon \mathrm{Idem}(R) \to \mathrm{Idem}(R').$$

Example **(1.12)**. — Let $R := R' \times R''$ be a **product** of two rings: its operations are performed componentwise. The additive identity is $(0, 0)$; the multiplicative identity is $(1, 1)$. Set $e := (1, 0)$ and $e' := (0, 1)$. Then e and e' are complementary idempotents. The next proposition shows this example is the only one possible.

Proposition **(1.13)**. — *Let R be a ring with complementary idempotents e and e'. Set $R' := Re$ and $R'' := Re'$, and form the map $\varphi\colon R \to R' \times R''$ defined by $\varphi(x) := (xe, xe')$. Then φ is a ring isomorphism.*

Proof. — Define a map $\varphi'\colon R \to R'$ by $\varphi'(x) := xe$. Then φ' is a ring map since $xye = xye^2 = (xe)(ye)$. Similarly, define $\varphi''\colon R \to R''$ by $\varphi''(x) := xe'$; then φ'' is a ring map. So φ is a ring map. Further, φ is surjective, since $(xe, x'e') = \varphi(xe + x'e')$. Also, φ is injective, since if $xe = 0$ and $xe' = 0$, then $x = xe + xe' = 0$. Thus φ is an isomorphism. □

Exercise (1.14) (*Chinese Remainder Theorem*). — Let R be a ring.

(1) Let $\mathfrak{a}$ and $\mathfrak{b}$ be **comaximal** ideals; that is, $\mathfrak{a} + \mathfrak{b} = R$. Prove

$$(a)\ \mathfrak{a}\mathfrak{b} = \mathfrak{a} \cap \mathfrak{b} \quad \text{and} \quad (\mathrm{b})\ R/\mathfrak{a}\mathfrak{b} = (R/\mathfrak{a}) \times (R/\mathfrak{b}).$$

(2) Let $\mathfrak{a}$ be comaximal to both $\mathfrak{b}$ and $\mathfrak{b}'$. Prove $\mathfrak{a}$ is also comaximal to $\mathfrak{b}\mathfrak{b}'$.

(3) Let $\mathfrak{a}$, $\mathfrak{b}$ be comaximal, and $m, n \ge 1$. Prove $\mathfrak{a}^m$ and $\mathfrak{b}^n$ are comaximal.

(4) Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be pairwise comaximal. Prove

(a) $\mathfrak{a}_1$ and $\mathfrak{a}_2 \cdots \mathfrak{a}_n$ are comaximal;
(b) $\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n = \mathfrak{a}_1 \cdots \mathfrak{a}_n$;
(c) $R/(\mathfrak{a}_1 \cdots \mathfrak{a}_n) \xrightarrow{\sim} \prod (R/\mathfrak{a}_i)$.

Exercise (1.15). — First, given a prime number p and a $k \ge 1$, find the idempotents in $\mathbb{Z}/\langle p^k \rangle$. Second, find the idempotents in $\mathbb{Z}/\langle 12 \rangle$. Third, find the number of idempotents in $\mathbb{Z}/\langle n \rangle$ where $n = \prod_{i=1}^{N} p_i^{n_i}$ with p_i distinct prime numbers.

Exercise (1.16). — Let $R := R' \times R''$ be a product of rings, $\mathfrak{a} \subset R$ an ideal. Show $\mathfrak{a} = \mathfrak{a}' \times \mathfrak{a}''$ with $\mathfrak{a}' \subset R'$ and $\mathfrak{a}'' \subset R''$ ideals. Show $R/\mathfrak{a} = (R'/\mathfrak{a}') \times (R''/\mathfrak{a}'')$.

(2.6) (*Unique factorization*). — Let R be a domain, p a nonzero nonunit. We call p **prime** if, whenever $p \mid xy$ (that is, there exists $z \in R$ such that $pz = xy$), either $p \mid x$ or $p \mid y$. Clearly, *p is prime if and only if the ideal $\langle p\rangle$ is prime.*

We call p **irreducible** if, whenever $p = yz$, either y or z is a unit. We call R a **Unique Factorization Domain** (UFD) if every nonzero element is a product of irreducible elements in a unique way up to order and units.

In general, prime elements are irreducible; in a UFD, irreducible elements are prime. Standard examples of UFDs include any field, the integers $\mathbb{Z}$, and a polynomial ring in n variables over a UFD; see [2, p. 398, p. 401], [8, Cor. 18.23, p. 297].

Lemma **(2.7)**. — *Let $\varphi\colon R \to R'$ be a ring map, and $T \subset R'$ a subset. If T is multiplicative, then $\varphi^{-1}T$ is multiplicative; the converse holds if φ is surjective.*

Proof: Set $S := \varphi^{-1}T$. If T is multiplicative, then $1 \in S$ as $\varphi(1) = 1 \in T$, and $x, y \in S$ implies $xy \in S$ as $\varphi(xy) = \varphi(x)\varphi(y) \in T$; thus S is multiplicative.

If S is multiplicative, then $1 \in T$ as $1 \in S$ and $\varphi(1) = 1$; further, $x, y \in S$ implies $\varphi(x)$, $\varphi(y)$, $\varphi(xy) \in T$. If φ is surjective, then every $x' \in T$ is of the form $x' = \varphi(x)$ for some $x \in S$. Thus if φ is surjective, then T is multiplicative if $\varphi^{-1}T$ is. □

Proposition **(2.8)**. — *Let $\varphi\colon R \to R'$ be a ring map, and $\mathfrak{q} \subset R'$ an ideal. If $\mathfrak{q}$ is prime, then $\varphi^{-1}\mathfrak{q}$ is prime; the converse holds if φ is surjective.*

Proof: By (2.7), $R - \mathfrak{p}$ is multiplicative if and only if $R' - \mathfrak{q}$ is. So the assertion results from Definitions (2.1). □

Corollary **(2.9)**. — *Let R be a ring, $\mathfrak{p}$ an ideal. Then $\mathfrak{p}$ is prime if and only if $R/\mathfrak{p}$ is a domain.*

Proof: By (2.8), $\mathfrak{p}$ is prime if and only if $\langle 0\rangle \subset R/\mathfrak{p}$ is. So the assertion results from the definition of domain in (2.3). □

Exercise **(2.10)**. — Let R be a domain, and $R[X_1, \ldots, X_n]$ the polynomial ring in n variables. Let $m \le n$, and set $\mathfrak{p} := \langle X_1, \ldots, X_m\rangle$. Prove $\mathfrak{p}$ is a prime ideal.

Exercise **(2.11)**. — Let $R := R' \times R''$ be a **product** of rings, $\mathfrak{p} \subset R$ an ideal. Show $\mathfrak{p}$ is prime if and only if either $\mathfrak{p} = \mathfrak{p}' \times R''$ with $\mathfrak{p}' \subset R'$ prime or $\mathfrak{p} = R' \times \mathfrak{p}''$ with $\mathfrak{p}'' \subset R''$ prime.

Exercise **(2.12)**. — Let R be a domain, and $x, y \in R$. Assume $\langle x\rangle = \langle y\rangle$. Show $x = uy$ for some unit u.

Definition **(2.13)**. — Let R be a ring. An ideal $\mathfrak{m}$ is said to be **maximal** if $\mathfrak{m}$ is proper and if there is no proper ideal $\mathfrak{a}$ with $\mathfrak{m} \subsetneqq \mathfrak{a}$.

Example **(2.14)**. — Let R be a domain. In the polynomial ring $R[X, Y]$ in two variables, $\langle X\rangle$ is prime by (2.10). However, $\langle X\rangle$ is not maximal since $\langle X\rangle \subsetneqq \langle X, Y\rangle$. Moreover, $\langle X, Y\rangle$ is maximal if and only if R is a field by (1.10) and by (2.17) below.

Proposition **(2.15)**. — *A ring R is a field if and only if $\langle 0\rangle$ is a maximal ideal.*

Proof: Suppose R is a field. Let $\mathfrak{a}$ be a nonzero ideal, and a a nonzero element of $\mathfrak{a}$. Since R is a field, $a \in R^\times$. So (1.4) yields $\mathfrak{a} = R$.

Conversely, suppose $\langle 0\rangle$ is maximal. Take $x \ne 0$. Then $\langle x\rangle \ne \langle 0\rangle$. So $\langle x\rangle = R$. So x is a unit by (1.4). Thus R is a field. □

3. Radicals

Two radicals of a ring are commonly used in Commutative Algebra: the Jacobson radical, which is the intersection of all maximal ideals, and the nilradical, which is the set of all nilpotent elements. Closely related to the nilradical is the radical of a subset. We define these three radicals, and discuss examples. In particular, we study local rings; a local ring has only one maximal ideal, which is then its Jacobson radical. We prove two important general results: *Prime Avoidance*, which states that, if an ideal lies in a finite union of primes, then it lies in one of them, and the *Scheinnullstellensatz*, which states that the nilradical of an ideal is equal to the intersection of all the prime ideals containing it.

Definition **(3.1)**. — Let R be a ring. Its (Jacobson) **radical** $\operatorname{rad}(R)$ is defined to be the intersection of all its maximal ideals.

Proposition **(3.2)**. — *Let R be a ring, $x \in R$, and $u \in R^\times$. Then $x \in \operatorname{rad}(R)$ if and only if $u - xy \in \operatorname{rad}(R)$ is a unit for all $y \in R$. In particular, the sum of an element of $\operatorname{rad}(R)$ and a unit is a unit.*

Proof: Assume $x \in \operatorname{rad}(R)$. Let $\mathfrak{m}$ be a maximal ideal. Suppose $u - xy \in \mathfrak{m}$. Since $x \in \mathfrak{m}$ too, also $u \in \mathfrak{m}$, a contradiction. Thus $u - xy$ is a unit by **(2.31)**. In particular, taking $y := -1$ yields $u + x \in R^\times$.

Conversely, assume $x \notin \operatorname{rad}(R)$. Then there is a maximal ideal $\mathfrak{m}$ with $x \notin \mathfrak{m}$. So $\langle x \rangle + \mathfrak{m} = R$. Hence there exist $y \in R$ and $m \in \mathfrak{m}$ such that $xy + m = u$. Then $u - xy = m \in \mathfrak{m}$. So $u - xy$ is not a unit by **(2.31)**, or directly by **(1.4)**. $\square$

Exercise **(3.3)**. — Let R be a ring, $\mathfrak{a} \subset \operatorname{rad}(R)$ an ideal, $w \in R$, and $w' \in R/\mathfrak{a}$ its residue. Prove that $w \in R^\times$ if and only if $w' \in (R/\mathfrak{a})^\times$. What if $\mathfrak{a} \not\subset \operatorname{rad}(R)$?

Corollary **(3.4)**. — *Let R be a ring, $\mathfrak{a}$ an ideal, $\kappa\colon R \to R/\mathfrak{a}$ the quotient map. Assume $\mathfrak{a} \subset \operatorname{rad}(R)$. Then $\operatorname{Idem}(\kappa)$ is injective.*

Proof: Given $e, e' \in \operatorname{Idem}(R)$ with $\kappa(e) = \kappa(e')$, set $x := e - e'$. Then

$$x^3 = e^3 - 3e^2e' + 3ee'^2 - e'^3 = e - e' = x.$$

Hence $x(1 - x^2) = 0$. But $\kappa(x) = 0$; so $x \in \mathfrak{a}$. But $\mathfrak{a} \subset \operatorname{rad}(R)$. Hence $1 - x^2$ is a unit by **(3.2)**. Thus $x = 0$. Thus $\operatorname{Idem}(\kappa)$ is injective. $\square$

Definition **(3.5)**. — A ring A is called **local** if it has exactly one maximal ideal, and **semilocal** if it has at least one and at most finitely many.

Lemma **(3.6)** (Nonunit Criterion). — *Let A be a ring, $\mathfrak{n}$ the set of nonunits. Then A is local if and only if $\mathfrak{n}$ is an ideal; if so, then $\mathfrak{n}$ is the maximal ideal.*

Proof: Every proper ideal $\mathfrak{a}$ lies in $\mathfrak{n}$ as $\mathfrak{a}$ contains no unit. So, if $\mathfrak{n}$ is an ideal, then it is a maximal ideal, and the only one. Thus A is local.

Conversely, assume A is local with maximal ideal $\mathfrak{m}$. Then $A - \mathfrak{n} = A - \mathfrak{m}$ by **(2.31)**. So $\mathfrak{n} = \mathfrak{m}$. Thus $\mathfrak{n}$ is an ideal. $\square$

Example **(3.7)**. — The product ring $R' \times R''$ is not local by **(3.6)** if both R' and R'' are nonzero. Indeed, $(1,0)$ and $(0,1)$ are nonunits, but their sum is a unit.

Exercise (4.19). — Let L be a module, Λ a nonempty set, M_λ a module for $\lambda \in \Lambda$. Prove that the injections $\iota_\kappa \colon M_\kappa \to \bigoplus M_\lambda$ induce an injection

$$\bigoplus \operatorname{Hom}(L, M_\lambda) \hookrightarrow \operatorname{Hom}(L, \bigoplus M_\lambda),$$

and that it is an isomorphism if L is finitely generated.

Exercise (4.20). — Let $\mathfrak{a}$ be an ideal, Λ a nonempty set, M_λ a module for $\lambda \in \Lambda$. Prove $\mathfrak{a}\big(\bigoplus M_\lambda\big) = \bigoplus \mathfrak{a}M_\lambda$. Prove $\mathfrak{a}(\prod M_\lambda) = \prod \mathfrak{a}M_\lambda$ if $\mathfrak{a}$ is finitely generated.

maximality, $\langle a_1\rangle = \gamma(N)$. But $\langle b\rangle \subset \langle c\rangle$. Thus $\beta(y_1) = b \in \langle a_1\rangle$.

Write $y_1 = \sum c_\lambda e_\lambda$ for some $c_\lambda \in R$. Then $\pi_\lambda(y_1) = c_\lambda$. But $c_\lambda = a_1 d_\lambda$ for some $d_\lambda \in R$ by the above paragraph with $\beta := \pi_\lambda$. Set $x_1 := \sum d_\lambda e_\lambda$. Then $y_1 = a_1 x_1$.

So $\alpha_1(y_1) = a_1\alpha_1(x_1)$. But $\alpha_1(y_1) = a_1$. So $a_1\alpha_1(x_1) = a_1$. But R is a domain and $a_1 \neq 0$. Thus $\alpha_1(x_1) = 1$.

Set $M_1 := \mathrm{Ker}(\alpha_1)$. As $\alpha_1(x_1) = 1$, clearly $Rx_1 \cap M_1 = 0$. Also, given $x \in M$, write $x = \alpha_1(x)x_1 + (x - \alpha_1(x)x_1)$; thus $x \in Rx_1 + M_1$. Hence **(4.17)** implies $M = Rx_1 \oplus M_1$. Further, M_1 is free by **(4.14)**. Set $N_1 := M_1 \cap N$.

Recall $a_1x_1 = y_1 \in N$. So $N \supset Ra_1x_1 \oplus N_1$. Conversely, given $y \in N$, write $y = bx_1 + m_1$ with $b \in R$ and $m_1 \in M_1$. Then $\alpha_1(y) = b$, so $b \in \langle a_1\rangle$. Hence $y \in Ra_1x_1 + N_1$. Thus $N = Ra_1x_1 \oplus N_1$.

Define $\varphi\colon R \to Ra_1x_1$ by $\varphi(a) = aa_1x_1$. If $\varphi(a) = 0$, then $aa_1 = 0$ as $\alpha_1(x_1) = 1$, and so $a = 0$ as $a_1 \neq 0$. Thus φ is injective, so a isomorphism.

Note $N_1 \simeq R^m$ with $m \leq n$ owing to **(4.14)** with N for E. Hence $N \simeq R^{m+1}$. But $N \simeq R^n$. So **(5.32)**(2) yields $m + 1 = n$.

By induction on n, there exists a decomposition $M_1 = M_1' \oplus M''$ and elements $x_2, \ldots, x_n \in M_1'$ and $a_2, \ldots, a_n \in R$ such that

$$M_1' = Rx_2 \oplus \cdots \oplus Rx_n,\ \ N_1 = Ra_2x_2 \oplus \cdots \oplus Ra_nx_n,\ \ \langle a_2\rangle \supset \cdots \supset \langle a_n\rangle \neq 0.$$

Then $M = M' \oplus M''$ and $M' = Rx_1 \oplus \cdots \oplus Rx_n$ and $N = Ra_1x_1 \oplus \cdots \oplus Ra_nx_n$. Also $\langle a_1\rangle \supset \cdots \supset \langle a_n\rangle \neq 0$. Thus existence is proved.

Finally, consider the projection $\pi\colon M_1 \to R$ with $\pi(x_j) = \delta_{2j}$ for $j \leq 2 \leq n$ and $\pi|M'' = 0$. Define $\rho\colon M \to R$ by $\rho(ax_1 + m_1) := a + \pi(m_1)$. Then $\rho(a_1x_1) = a_1$. So $\rho(N) \supset \langle a_1\rangle = \alpha_1(N)$. By maximality, $\rho(N) = \alpha_1(N)$. But $a_2 = \rho(a_2x_2) \in \rho(N)$. Thus $\langle a_2\rangle \subset \langle a_1\rangle$ as desired.

Moreover, $M' = \{m \in M \mid xm \in N \text{ for some } x \in R\}$. Thus M' is determined. Also, by **(5.37)**(2) with M'/N for M, each a_i is determined up to unit multiple. □

Theorem **(5.39)**. — *Let A be a local ring, M a finitely presented module.*

(1) *Then M can be generated by m elements if and only if $F_m(M) = A$.*

(2) *Then M is free of rank m if and only if $F_m(M) = A$ and $F_{m-1}(M) = \langle 0\rangle$.*

Proof: For (1), assume M can be generated by m elements. Then **(4.10)**(1) and **(5.26)** yield a presentation $A^n \xrightarrow{\alpha} A^m \to M \to 0$. So $F_m(M) = A$ by **(5.34)**.

For the converse, assume also M cannot be generated by $m-1$ elements. Suppose $F_k(M) = A$ with $k < m$. Then $F_{m-1}(M) = A$ by **(5.35.1)**. Hence one entry of the matrix (a_{ij}) of α does not belong to the maximal ideal, so is a unit by **(3.6)**. By **(5.33)**(2), we may assume $a_{11} = 1$ and the other entries in the first row and first column of $\mathbf{A}$ are 0. Thus $\mathbf{A} = \left(\begin{smallmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}\end{smallmatrix}\right)$ where $\mathbf{B}$ is an $(m-1)\times(s-1)$ matrix. Then $\mathbf{B}$ defines a presentation $A^{s-1} \to A^{m-1} \to M \to 0$. So M can be generated by $m-1$ elements, a contradiction. Thus $F_k(M) \neq A$ for $k < m$. Thus (1) holds.

In (2), if M is free of rank m, then there's a presentation $0 \to A^m \to M \to 0$; so $F_m(M) = A$ and $F_{m-1}(M) = \langle 0\rangle$ by **(5.35)**. Conversely, if $F_m(M) = A$, then (1) and **(5.26)** and **(4.10)**(1) yield a presentation $A^s \xrightarrow{\alpha} A^m \to M \to 0$. If also $F_{m-1}(M) = \langle 0\rangle$, then $\alpha = 0$ by **(5.35)**. Thus M is free of rank m; so (2) holds. □

Proposition **(5.40)**. — *Let R be a ring, and M a finitely presented module. Say M can be generated by m elements. Set $\mathfrak{a} := \mathrm{Ann}(M)$. Then*

$$(1)\ \ \mathfrak{a}F_r(M) \subset F_{r-1}(M) \text{ for all } r > 0 \quad \text{and} \quad (2)\ \ \mathfrak{a}^m \subset F_0(M) \subset \mathfrak{a}.$$

Proof: As M can be generated by m elements, **(4.10)**(1) and **(5.26)** yield a presentation $A^n \xrightarrow{\alpha} A^m \xrightarrow{\mu} M \to 0$. Say α has matrix $\mathbf{A}$.

In (1), if $r > m$, then trivially $\mathfrak{a}F_r(M) \subset F_{r-1}(M)$ owing to **(5.35.1)**. So assume $r \le m$ and set $s := m - r + 1$. Given $x \in \mathfrak{a}$, form the sequence

$$R^{n+m} \xrightarrow{\beta} R^m \xrightarrow{\mu} M \to 0 \text{ with } \beta := \alpha + x1_{R^m}.$$

Note that this sequence is a presentation. Also, the matrix of β is $(\mathbf{A}|x\mathbf{I}_m)$, obtained by juxtaposition, where $\mathbf{I}_m$ is the $m \times m$ identity matrix.

Given an $(s-1) \times (s-1)$ submatrix $\mathbf{B}$ of $\mathbf{A}$, enlarge it to an $s \times s$ submatrix $\mathbf{B}'$ of $(\mathbf{A}|x\mathbf{I}_m)$ as follows: say the ith row of $\mathbf{A}$ is not involved in $\mathbf{B}$; form the $m \times s$ submatrix $\mathbf{B}''$ of $(\mathbf{A}|x\mathbf{I}_m)$ with the same columns as $\mathbf{B}$ plus the ith column of $x\mathbf{I}_m$ at the end; finally, form $\mathbf{B}'$ as the $s \times s$ submatrix of $\mathbf{B}''$ with the same rows as $\mathbf{B}$ plus the ith row in the appropriate position.

Expanding along the last column yields $\det(\mathbf{B}') = \pm x \det(\mathbf{B})$. By constuction, $\det(\mathbf{B}') \in I_s(\mathbf{A}|x\mathbf{I}_m)$. But $I_s(\mathbf{A}|x\mathbf{I}_m) = I_s(\mathbf{A})$ by **(5.34)**. Furthermore, $x \in \mathfrak{a}$ is arbitrary, and $I_m(\mathbf{A})$ is generated by all possible $\det(\mathbf{B})$. Thus (1) holds.

For (2), apply (1) repeatedly to get $\mathfrak{a}^k F_r(M) \subset F_{r-k}(M)$ for all r and k. But $F_m(M) = R$ by **(5.35.1)**. So $\mathfrak{a}^m \subset F_0(M)$.

For the second inclusion, given any $m \times m$ submatrix $\mathbf{B}$ of $\mathbf{A}$, say $\mathbf{B} = (b_{ij})$. Let $\mathbf{e}_i$ be the ith standard basis vector of R^m. Set $m_i := \mu(\mathbf{e}_i)$. Then $\sum b_{ij} m_j = 0$ for all i. Let $\mathbf{C}$ be the matrix of cofactors of $\mathbf{B}$: the (i,j)th entry of $\mathbf{C}$ is $(-1)^{i+j}$ times the determinant of the matrix obtained by deleting the jth row and the ith column of $\mathbf{B}$. Then $\mathbf{CB} = \det(\mathbf{B})\mathbf{I}_m$. Hence $\det(\mathbf{B})m_i = 0$ for all i. So $\det(\mathbf{B}) \in \mathfrak{a}$. But $I_m(\mathbf{A})$ is generated by all such $\det(\mathbf{B})$. Thus $F_0(M) \subset \mathfrak{a}$. Thus (2) holds. □

(6.6) (*Direct limits*). — Let Λ, $\mathcal{C}$ be categories. Assume Λ is **small**; that is, its objects form a set. Given a functor $\lambda \mapsto M_\lambda$ from Λ to $\mathcal{C}$, its **direct limit** or **colimit**, denoted $\varinjlim M_\lambda$ or $\varinjlim_{\lambda\in\Lambda} M_\lambda$, is defined to be the object of $\mathcal{C}$ universal among objects P equipped with maps $\beta_\mu\colon M_\mu \to P$, called **insertions**, that are compatible with the **transition maps** $\alpha^\kappa_\mu\colon M_\kappa \to M_\mu$, which are the images of the maps of Λ. (Note: given κ and μ, there may be more than one map $\kappa \to \mu$, and so more than one transition map α^κ_μ.) In other words, there is a unique map β such that all of the following diagrams commute:

$$\begin{tikzcd} M_\kappa \arrow[r, "\alpha^\kappa_\mu"] \arrow[d, "\beta_\kappa"] & M_\mu \arrow[r, "\alpha_\mu"] \arrow[d, "\beta_\mu"] & \varinjlim M_\lambda \arrow[d, "\beta"] \\ P \arrow[r, "1_P"] & P \arrow[r, "1_P"] & P \end{tikzcd}$$

To indicate this context, the functor $\lambda \mapsto M_\lambda$ is often called a **direct system.**

As usual, universality implies that, once equipped with its insertions α_μ, the limit $\varinjlim M_\lambda$ is determined up to unique isomorphism, assuming it exists. In practice, there is usually a canonical choice for $\varinjlim M_\lambda$, given by a construction. In any case, let us use $\varinjlim M_\lambda$ to denote a particular choice.

We say that $\mathcal{C}$ **has direct limits indexed by** Λ if, for every functor $\lambda \mapsto M_\lambda$ from Λ to $\mathcal{C}$, the direct limit $\varinjlim M_\lambda$ exists. We say that $\mathcal{C}$ **has direct limits** if it has direct limits indexed by every small category Λ.

Given a functor $F\colon \mathcal{C} \to \mathcal{C}'$, note that a functor $\lambda \mapsto M_\lambda$ from Λ to $\mathcal{C}$ yields a functor $\lambda \mapsto F(M_\lambda)$ from Λ to $\mathcal{C}'$. Furthermore, whenever the corresponding two direct limits exist, the maps $F(\alpha_\mu)\colon F(M_\mu) \to F(\varinjlim M_\lambda)$ induce a canonical map

$$\phi\colon \varinjlim F(M_\lambda) \to F(\varinjlim M_\lambda). \tag{6.6.1}$$

If ϕ is always an isomorphism, we say F **preserves direct limits**. At times, given $\varinjlim M_\lambda$, we construct $\varinjlim F(M_\lambda)$ by showing $F(\varinjlim M_\lambda)$ has the requisite UMP.

Assume $\mathcal{C}$ has direct limits indexed by Λ. Then, given a natural transformation from $\lambda \mapsto M_\lambda$ to $\lambda \mapsto N_\lambda$, universality yields unique commutative diagrams

$$\begin{tikzcd} M_\mu \arrow[r] \arrow[d] & \varinjlim M_\lambda \arrow[d] \\ N_\mu \arrow[r] & \varinjlim N_\lambda \end{tikzcd}$$

To put it in another way, form the **functor category** $\mathcal{C}^\Lambda$: its objects are the functors $\lambda \mapsto M_\lambda$ from Λ to $\mathcal{C}$; its maps are the natural transformations (they form a set as Λ is one). Then taking direct limits yields a functor $\varinjlim$ from $\mathcal{C}^\Lambda$ to $\mathcal{C}$.

In fact, it is just a restatement of the definitions that the "direct limit" functor $\varinjlim$ is the left adjoint of the **diagonal functor**

$$\Delta\colon \mathcal{C} \to \mathcal{C}^\Lambda.$$

By definition, Δ sends each object M to the **constant functor** ΔM, which has the same value M at every $\lambda \in \Lambda$ and has the same value 1_M at every map of Λ; further, Δ carries a map $\gamma\colon M \to N$ to the natural transformation $\Delta\gamma\colon \Delta M \to \Delta N$, which has the same value γ at every $\lambda \in \Lambda$.

(6.7) (*Coproducts*). — Let $\mathcal{C}$ be a category, Λ a set, and M_λ an object of $\mathcal{C}$ for each $\lambda \in \Lambda$. The **coproduct** $\coprod_{\lambda\in\Lambda} M_\lambda$, or simply $\coprod M_\lambda$, is defined as the object of $\mathcal{C}$ universal among objects P equipped with a map $\beta_\mu\colon M_\mu \to P$ for each $\mu \in \Lambda$.

Proof: The assertions follow directly from **(7.7)**. Specifically, (1) holds, since $\varinjlim M_\lambda$ is a quotient of the disjoint union $\bigsqcup M_\lambda$. Further, (2) holds owing to the definition of the equivalence relation involved. Finally, (3) is the special case of (2) where $m_1 := m_\lambda$ and $m_2 = 0$. $\square$

Exercise **(7.9)**. — Let $R := \varinjlim R_\lambda$ be a filtered direct limit of rings.
(1) Prove that $R = 0$ if and only if $R_\lambda = 0$ for some λ.
(2) Assune that each R_λ is a domain. Prove that R is a domain.
(3) Assume that each R_λ is a field. Prove that R is a field.

Exercise **(7.10)**. — Let $M := \varinjlim M_\lambda$ be a filtered direct limit of modules, with transition maps $\alpha^\lambda_\mu\colon M_\lambda \to M_\mu$ and insertions $\alpha_\lambda\colon M_\lambda \to M$. For each λ, let $N_\lambda \subset M_\lambda$ be a submodule, and let $N \subset M$ be a submodule. Prove that $N_\lambda = \alpha_\lambda^{-1}N$ for all λ if and only if (a) $N_\lambda = (\alpha^\lambda_\mu)^{-1}N_\mu$ for all α^λ_μ and (b) $\bigcup \alpha_\lambda N_\lambda = N$.

Definition **(7.11)**. — Let R be a ring. We say an algebra R' is *finitely presented* if $R' \simeq R[X_1, \ldots, X_r]/\mathfrak{a}$ for some variables X_i and finitely generated ideal $\mathfrak{a}$.

Proposition **(7.12)**. — *Let Λ be a filtered category, R a ring, $\mathcal{C}$ either $((R\text{-mod}))$ or $((R\text{-alg}))$, $\lambda \mapsto M_\lambda$ a functor from Λ to $\mathcal{C}$. Given $N \in \mathcal{C}$, form the map* (6.6.1),

$$\theta\colon \varinjlim \mathrm{Hom}(N, M_\lambda) \to \mathrm{Hom}(N, \varinjlim M_\lambda).$$

(1) *If N is finitely generated, then θ is injective.*
(2) *The following conditions are equivalent:*
(a) *N is finitely presented;*
(b) *θ is bijective for all filtered categories Λ and all functors $\lambda \mapsto M_\lambda$;*
(c) *θ is surjective for all directed sets Λ and all $\lambda \mapsto M_\lambda$.*

Proof: Given a transition map $\alpha^\lambda_\mu\colon M_\lambda \to M_\mu$, set $\beta^\lambda_\mu := \mathrm{Hom}(N, \alpha^\lambda_\mu)$. Then the β^λ_μ are the transition maps of $\varinjlim \mathrm{Hom}(N, M_\lambda)$. Denote by α_λ and β_λ the insertions of $\varinjlim M_\lambda$ and $\varinjlim \mathrm{Hom}(N, M_\lambda)$.

For (1), let $n_1, \ldots, n_r$ generate N. Given φ and φ' in $\varinjlim \mathrm{Hom}(N, M_\lambda)$ with $\theta(\varphi) = \theta(\varphi')$, note that **(7.8)**(1) yields λ and $\varphi_\lambda\colon N \to M_\lambda$ and μ and $\varphi'_\mu\colon N \to M_\mu$ with $\beta_\lambda(\varphi_\lambda) = \varphi$ and $\beta_\mu(\varphi'_\mu) = \varphi'$. Then $\theta(\varphi) = \alpha_\lambda\varphi_\lambda$ and $\theta(\varphi') = \alpha_\mu\varphi'_\mu$ by construction of θ. Hence $\alpha_\lambda\varphi_\lambda = \alpha_\mu\varphi'_\mu$. So $\alpha_\lambda\varphi_\lambda(n_i) = \alpha_\mu\varphi'_\mu(n_i)$ for all i. So **(7.8)**(2) yields λ_i and $\alpha^\lambda_{\lambda_i}$ and $\alpha^\mu_{\lambda_i}$ such that $\alpha^\lambda_{\lambda_i}\varphi_\lambda(n_i) = \alpha^\mu_{\lambda_i}\varphi'_\mu(n_i)$ for all i.

Let's prove, by induction on i, that there are ν_i and $\alpha^\lambda_{\nu_i}$ and $\alpha^\mu_{\nu_i}$ such that $\alpha^\lambda_{\nu_i}\varphi_\lambda(n_j) = \alpha^\mu_{\nu_i}(n_j)$ for $1 \le j \le i$. Indeed, given ν_{i-1} and $\alpha^\lambda_{\nu_{i-1}}$ and $\alpha^\mu_{\nu_{i-1}}$, by **(7.1)**(1), there are ρ_i and $\alpha^{\nu_{i-1}}_{\rho_i}$ and $\alpha^{\lambda_i}_{\rho_i}$. By **(7.1)**(2), there are ν_i and $\alpha^{\rho_i}_{\nu_i}$ such that $\alpha^{\rho_i}_{\nu_i}\alpha^{\nu_{i-1}}_{\rho_i}\alpha^\lambda_{\nu_{i-1}} = \alpha^{\rho_i}_{\nu_i}\alpha^{\lambda_i}_{\rho_i}\alpha^\lambda_{\lambda_i}$ and $\alpha^{\rho_i}_{\nu_i}\alpha^{\nu_{i-1}}_{\rho_i}\alpha^\mu_{\nu_{i-1}} = \alpha^{\rho_i}_{\nu_i}\alpha^{\lambda_i}_{\rho_i}\alpha^\mu_{\lambda_i}$. Set $\alpha^\lambda_{\nu_i} := \alpha^{\rho_i}_{\nu_i}\alpha^{\lambda_i}_{\rho_i}\alpha^\lambda_{\lambda_i}$ and $\alpha^\mu_{\nu_i} := \alpha^{\rho_i}_{\nu_i}\alpha^{\lambda_i}_{\rho_i}\alpha^\mu_{\lambda_i}$. Then $\alpha^\lambda_{\nu_i}\varphi_\lambda(n_j) = \alpha^\mu_{\nu_i}(n_j)$ for $1 \le j \le i$, as desired.

Set $\nu := \nu_r$. Then $\alpha^\lambda_\nu\varphi_\lambda(n_i) = \alpha^\mu_\nu\varphi'_\mu(n_i)$ for all i. Hence $\alpha^\lambda_\nu\varphi_\lambda = \alpha^\mu_\nu\varphi'_\mu$. But

$$\varphi = \beta_\lambda(\varphi_\lambda) = \beta_\nu\beta^\lambda_\nu(\varphi_\lambda) = \beta_\nu(\alpha^\lambda_\nu\varphi_\lambda).$$

Similarly, $\varphi' = \beta_\nu(\alpha^\mu_\nu\varphi'_\mu)$. Hence $\varphi = \varphi'$. Thus θ is injective. Notice that this proof works equally well for $((R\text{-mod}))$ and $((R\text{-alg}))$. Thus (1) holds.

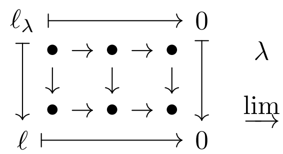
For (2), let's treat the case $\mathcal{C} = ((R\text{-mod}))$ first. Assume (a). Say $N \simeq F/N'$ where $F := R^r$ and N' is finitely generated, say by $n'_1, \ldots, n'_s$. Let n_i be the image in N of the ith standard basis vector e_i of F. Then there are homogeneous linear polynomials f_j with $f_j(e_1, \ldots, e_r) = n'_j$ for all j. So $f_j(n_1, \ldots, n_r) = 0$.

objects are the 3-term exact sequences, and its maps are the commutative diagrams

$$\begin{tikzcd} L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[d] \\ L' \arrow[r] & M' \arrow[r] & N' \end{tikzcd}$$

Then, for any functor $\lambda \mapsto (L_\lambda \xrightarrow{\beta_\lambda} M_\lambda \xrightarrow{\gamma_\lambda} N_\lambda)$ *from* Λ *to* $\mathfrak{C}$*, the induced sequence* $\varinjlim L_\lambda \xrightarrow{\beta} \varinjlim M_\lambda \xrightarrow{\gamma} \varinjlim N_\lambda$ *is exact.*

Proof: Abusing notation, in all three cases denote by α_λ^κ the transition maps and by α_λ the insertions. Then given $\ell \in \varinjlim L_\lambda$, there is $\ell_\lambda \in L_\lambda$ with $\alpha_\lambda \ell_\lambda = \ell$ by (7.8)(1). By hypothesis, $\gamma_\lambda\beta_\lambda\ell_\lambda = 0$; so $\gamma\beta\ell = 0$. In sum, we have this figure:

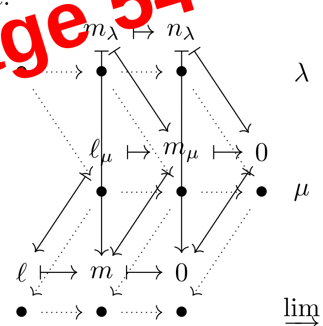


Thus $\mathrm{Im}(\beta) \subset \mathrm{Ker}(\gamma)$.

For the opposite inclusion, take $m \in \varinjlim M_\lambda$ with $\gamma m = 0$. By (7.8)(1) there is $m_\lambda \in M_\lambda$ with $\alpha_\lambda m_\lambda = m$. Now, $\alpha_\lambda\gamma_\lambda m_\lambda = 0$ by commutativity. So by (7.8)(3), there is α_μ^λ with $\alpha_\mu^\lambda\gamma_\lambda m_\lambda = 0$. So $\gamma_\mu\alpha_\mu^\lambda m_\lambda = 0$ by commutativity. Hence there is $\ell_\mu \in L_\mu$ with $\beta_\mu\ell_\mu = \alpha_\mu^\lambda m_\lambda$ by exactness. Apply α_μ to get

$$\beta\alpha_\mu\ell_\mu = \alpha_\mu\beta_\mu\ell_\mu = \alpha_\mu\alpha_\mu^\lambda m_\lambda = m.$$

In sum, we have this figure:



Thus $\mathrm{Ker}(\gamma) \subset \mathrm{Im}(\beta)$. So $\mathrm{Ker}(\gamma) = \mathrm{Im}(\beta)$ as asserted. □

Exercise (7.15). — Let $R := \varinjlim R_\lambda$ be a filtered direct limit of rings, $\mathfrak{a}_\lambda \subset R_\lambda$ an ideal for each λ. Assume $\alpha_\mu^\lambda\mathfrak{a}_\lambda \subset \mathfrak{a}_\mu$ for each transition map α_μ^λ. Set $\mathfrak{a} := \varinjlim \mathfrak{a}_\lambda$. If each $\mathfrak{a}_\lambda$ is prime, show $\mathfrak{a}$ is prime. If each $\mathfrak{a}_\lambda$ is maximal, show $\mathfrak{a}$ is maximal.

Exercise (7.16). — Let $M := \varinjlim M_\lambda$ be a filtered direct limit of modules, with transition maps $\alpha_\mu^\lambda \colon M_\lambda \to M_\mu$ and insertions $\alpha_\lambda \colon M_\lambda \to M$. Let $N_\lambda \subset M_\lambda$ be a be a submodule for all λ. Assume $\alpha_\mu^\lambda N_\lambda \subset N_\mu$ for all α_μ^λ. Prove $\varinjlim N_\lambda = \bigcup \alpha_\lambda N_\lambda$.

Exercise (7.17). — Let $R := \varinjlim R_\lambda$ be a filtered direct limit of rings. Prove that

$$\varinjlim \mathrm{nil}(R_\lambda) = \mathrm{nil}(R).$$

Exercise (**7.18**). — Let $R := \varinjlim R_\lambda$ be a filtered direct limit of rings. Assume each ring R_λ is local, say with maximal ideal $\mathfrak{m}_\lambda$, and assume each transition map $\alpha^\lambda_\mu \colon R_\lambda \to R_\mu$ is local. Set $\mathfrak{m} := \varinjlim \mathfrak{m}_\lambda$. Prove that R is local with maximal ideal $\mathfrak{m}$ and that each insertion $\alpha_\lambda \colon R_\lambda \to R$ is local.

(7.19) (*Hom and direct limits again*). — Let Λ a filtered category, R a ring, N a module, and $\lambda \mapsto M_\lambda$ a functor from Λ to $((R\text{-mod}))$. Here is an alternative proof that the map $\theta(N)$ of (**6.6.1**) is injective if N is finitely generated and bijective if N is finitely presented.

If $N := R$, then $\theta(N)$ is bijective by (**4.3**). Assume N is finitely generated, and take a presentation $R^{\oplus\Sigma} \to R^n \to N \to 0$ with Σ finite if N is finitely presented. It induces the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim \operatorname{Hom}(N, M_\lambda) \arrow[r] \arrow[d, "\theta(N)"'] & \varinjlim \operatorname{Hom}(R^n, M_\lambda) \arrow[r] \arrow[d, "\theta(R^n)"', "\simeq"] & \varinjlim \operatorname{Hom}(R^{\oplus\Sigma}, M_\lambda) \arrow[d, "\theta(R^{\oplus\Sigma})"'] \\
0 \arrow[r] & \operatorname{Hom}(N, \varinjlim M_\lambda) \arrow[r] & \operatorname{Hom}(R^n, \varinjlim M_\lambda) \arrow[r] & \operatorname{Hom}(R^{\oplus\Sigma}, \varinjlim M_\lambda)
\end{tikzcd}$$

The rows are exact owing to (**5.18**), the left exactness of Hom, and to (**7.14**), the exactness of filtered direct limits. Now, Hom preserves finite direct sums by (**4.15**), and direct limit does so by (**6.15**) and (**6.7**); hence, $\theta(R^n)$ is bijective, and $\theta(R^{\oplus\Sigma})$ is bijective if Σ is finite. A diagram chase yields the assertions.

Exercise (**7.20**). — Let Λ and Λ' be small categories, $C \colon \Lambda' \to \Lambda$ a functor. Assume Λ' is filtered. Assume C is **cofinal**; that is,

(1) given $\lambda \in \Lambda$, there is a map $\lambda \to C\lambda'$ for some $\lambda' \in \Lambda'$, and

(2) given $\psi, \varphi \colon \lambda \rightrightarrows C\lambda'$, there is $\chi \colon \lambda' \to \lambda'_1$ with $(C\chi)\psi = (C\chi)\varphi$.

Let $\lambda \mapsto M_\lambda$ be a functor from Λ to $\mathcal{C}$ whose direct limit exists. Show that

$$\varinjlim_{\lambda' \in \Lambda'} M_{C\lambda'} = \varinjlim_{\lambda \in \Lambda} M_\lambda;$$

more precisely, show that the right side has the UMP characterizing the left.

Exercise (**7.21**). — Show that every R-module M is the filtered direct limit over a directed set of finitely presented modules.

(8.5) (*Bifunctoriality*). — Let R be a ring, $\alpha\colon M \to M'$ and $\alpha'\colon N \to N'$ module homomorphisms. Then there is a canonical commutative diagram:

$$\begin{array}{ccc} M \times N & \xrightarrow{\alpha\times\alpha'} & M' \times N' \\ \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \beta'} \\ M \otimes N & \xrightarrow{\alpha\otimes\alpha'} & M' \otimes N' \end{array}$$

Indeed, $\beta' \circ (\alpha \times \alpha')$ is clearly bilinear; so the UMP **(8.3)** yields $\alpha \otimes \alpha'$. Thus $\bullet \otimes N$ and $M \otimes \bullet$ are commuting **linear** functors — that is, linear on maps, compare **(9.2)**.

Proposition **(8.6)**. — *Let R be a ring, M and N modules.*

(1) *Then the switch map $(m, n) \mapsto (n, m)$ induces an isomorphism*

$$M \otimes_R N = N \otimes_R M. \qquad \textbf{(commutative law)}$$

(2) *Then multiplication of R on M induces an isomorphism*

$$R \otimes_R M = M. \qquad \textbf{(unitary law)}$$

Proof: The switch map induces an isomorphism $R^{\oplus(M\times N)} \xrightarrow{\sim} R^{\oplus(N\times M)}$, and it preserves the elements of **(8.2.1)**. Thus (1) holds.

Define $\beta\colon R \times M \to M$ by $\beta(x, m) := xm$. Clearly β is bilinear. Let's check β has the requisite UMP. Given a bilinear map $\alpha\colon R \times M \to P$, define $\gamma\colon M \to P$ by $\gamma(m) := \alpha(1, m)$. Then γ is linear as α is bilinear. Also, $\alpha = \gamma\beta$ as

$$\alpha(x, m) = x\alpha(1, m) = \alpha(1, xm) = \gamma(xm) = \gamma\beta(x, m).$$

Further, γ is unique as β is surjective. Thus b has the UMP, so (2) holds. $\square$

Exercise **(8.7)**. — Let R be a domain, $\mathfrak{a}$ a nonzero ideal. Set $K := \mathrm{Frac}(R)$. Show that $\mathfrak{a} \otimes_R K = K$.

(8.8) (*Bimodules*). — Let R and R' be rings. An abelian group N is an (R, R')-**bimodule** if it is both an R-module and an R'-module and if $x(x'n) = x'(xn)$ for all $x \in R$, all $x' \in R'$, and all $n \in N$. At times, we think of N as a left R-module, with multiplication xn, and as a right R'-module, with multiplication nx'. Then the compatibility condition becomes the associative law: $x(nx') = (xn)x'$. A (R, R')-**homomorphism** of bimodules is a map that is both R-linear and R'-linear.

Let M be an R-module, and let N be an (R, R')-bimodule. Then $M \otimes_R N$ is an (R, R')-bimodule with R-structure as usual and with R'-structure defined by $x'(m \otimes n) := m \otimes (x'n)$ for all $x' \in R'$, all $m \in M$, and all $n \in N$. The latter multiplication is well defined and the two multiplications commute because of bifunctoriality **(8.5)** with $\alpha := \mu_x$ and $\alpha' := \mu_{x'}$.

For instance, suppose R' is an R-algebra. Then R' is an (R, R')-bimodule. So $M \otimes_R R'$ is an R'-module. It is said to be obtained by **extension of scalars**.

In full generality, it is easy to check that $\mathrm{Hom}_R(M, N)$ is an (R, R')-bimodule under valuewise multiplication by elements of R'. Further, given an R'-module P, it is easy to check that $\mathrm{Hom}_{R'}(N, P)$ is an (R, R')-bimodule under sourcewise multiplication by elements of R.

Exercise **(8.9)**. — Let R be a ring, R' an R-algebra, M, N two R'-modules. Show there is a canonical R-linear map $\tau\colon M \otimes_R N \to M \otimes_{R'} N$.

Let $K \subset M \otimes_R N$ denote the R-submodule generated by all the differences $(x'm) \otimes n - m \otimes (x'n)$ for $x' \in R'$ and $m \in M$ and $n \in N$. Show K is equal to $\mathrm{Ker}(\tau)$, and τ is surjective. Show τ is an isomorphism if R' is a quotient of R.

Theorem **(8.10)**. — *Let R and R' be rings, M an R-module, P an R'-module, N an (R, R')-bimodule. Then there are two canonical (R, R')-isomorphisms:*

$$M \otimes_R (N \otimes_{R'} P) = (M \otimes_R N) \otimes_{R'} P, \qquad \textbf{(associative law)}$$

$$\mathrm{Hom}_{R'}(M \otimes_R N,\, P) = \mathrm{Hom}_R\big(M,\ \mathrm{Hom}_{R'}(N, P)\big). \qquad \textbf{(adjoint associativity)}$$

Proof: Note that $M \otimes_R (N \otimes_{R'} P)$ and $(M \otimes_R N) \otimes_{R'} P$ are (R, R')-bimodules. For each (R, R')-bimodule Q, call a map $\tau\colon M \times N \times P \to Q$ **trilinear** if it is R-bilinear in $M \times N$ and R'-bilinear in $N \times P$. Denote the set of all these τ by $\mathrm{Tril}(M, N, P;\ Q)$. It is, clearly, an (R, R')-bimodule.

A trilinear map τ yields an R-bilinear map $M \times (N \otimes_{R'} P) \to Q$, whence a map $M \otimes_R (N \otimes_{R'} P) \to Q$, which is both R-linear and R'-linear, and *vice versa*. Thus

$$\mathrm{Tril}_{(R,R')}(M, N, P;\ Q) = \mathrm{Hom}\big(M \otimes_R (N \otimes_{R'} P),\ Q\big).$$

Similarly, there is a canonical isomorphism of (R, R')-bimodules

$$\mathrm{Tril}_{(R,R')}(M, N, P;\ Q) = \mathrm{Hom}\big((M \otimes_R N) \otimes_{R'} P,\ Q\big).$$

Hence each of $M \otimes_R (N \otimes_{R'} P)$ and $(M \otimes_R N) \otimes_{R'} P$ is the universal target of a trilinear map with source $M \times N \times P$. Thus they are equal, as asserted.

To establish the isomorphism of adjoint associativity, define a map

$$\alpha\colon \mathrm{Hom}_{R'}(M \otimes_R N, P) \to \mathrm{Hom}_R\big(M,\ \mathrm{Hom}_{R'}(N, P)\big) \quad \text{by}$$

$$\big(\alpha(\gamma)(m)\big)(n) := \gamma(m \otimes n).$$

Let's check α is well defined. First, $\alpha(\gamma)(m)$ is R'-linear, because given $x' \in R'$,

$$\gamma\big(m \otimes (x'n)\big) = \gamma\big(x'(m \otimes n)\big) = x'\gamma(m \otimes n)$$

since γ is R'-linear. Further, $\alpha(\gamma)$ is R-linear, because given $x \in R$,

$$(xm) \otimes n = m \otimes (xn) \quad \text{and so} \quad \big(\alpha(\gamma)(xm)\big)(n) = \big(\alpha(\gamma)(m)\big)(xn).$$

Thus $\alpha(\gamma) \in \mathrm{Hom}_R\big(M, \mathrm{Hom}_{R'}(N, P)\big)$. Clearly, α is an (R, R')-homomorphism.

To obtain an inverse to α, given $\eta \in \mathrm{Hom}_R\big(M, \mathrm{Hom}_{R'}(N, P)\big)$, define a map $\zeta\colon M \times N \to P$ by $\zeta(m, n) := (\eta(m))(n)$. Clearly, ζ is $\mathbb{Z}$-bilinear, so ζ induces a $\mathbb{Z}$-linear map $\delta\colon M \otimes_{\mathbb{Z}} N \to P$. Given $x \in R$, clearly $(\eta(xm))(n) = (\eta(m))(xn)$; so $\delta((xm) \otimes n) = \delta(m \otimes (xn))$. Hence, δ induces a $\mathbb{Z}$-linear map $\beta(\eta)\colon M \otimes_R N \to P$ owing to **(8.9)** with $\mathbb{Z}$ for R and with R for R'. Clearly, $\beta(\eta)$ is R'-linear as $\eta(m)$ is so. Finally, it is easy to verify that $\alpha(\beta(\eta)) = \eta$ and $\beta(\alpha(\gamma)) = \gamma$, as desired. □

Corollary **(8.11)**. — *Let R be a ring, and R' an algebra. First, let M be an R-module, and P an R'-module. Then there are two canonical R'-isomorphisms:*

$$(M \otimes_R R') \otimes_{R'} P = M \otimes_R P, \qquad \textbf{(cancellation law)}$$

$$\mathrm{Hom}_{R'}(M \otimes_R R',\, P) = \mathrm{Hom}_R(M,\, P). \qquad \textbf{(left adjoint)}$$

Instead, let M be an R'-module, and P an R-module. Then there is a canonical R'-isomorphism:

$$\mathrm{Hom}_R(M,\, P) = \mathrm{Hom}_{R'}(M,\ \mathrm{Hom}_R(R',\, P)). \qquad \textbf{(right adjoint)}$$

In other words, $\bullet \otimes_R R'$ is the left adjoint of restriction of scalars from R' to R, and $\mathrm{Hom}_R(R', \bullet)$ is its right adjoint.

Proof: The cancellation law results from the associative and unitary laws; the adjoint isomorphisms, from adjoint associativity, **(4.3)** and the unitary law. □

Exercise (**9.18**). — Prove that an R-algebra R' is faithfully flat if and only if the structure map $\varphi\colon R \to R'$ is injective and the quotient $R'/\varphi R$ is flat over R.

Proposition (**9.19**). — *A filtered direct limit of flat modules $\varinjlim M_\lambda$ is flat.*

Proof: Let $\beta\colon N' \to N$ be injective. Then $M_\lambda \otimes \beta$ is injective for each λ since M_λ is flat. So $\varinjlim(M_\lambda \otimes \beta)$ is injective by the exactness of filtered direct limits, (**7.14**). So $(\varinjlim M_\lambda) \otimes \beta$ is injective by (**8.13**). Thus $\varinjlim M_\lambda$ is flat. □

Proposition (**9.20**). — *Let R and R' be rings, M an R-module, N an (R, R')-bimodule, and P an R'-module. Then there is a canonical homomorphism*

$$\theta\colon \operatorname{Hom}_R(M,\, N) \otimes_{R'} P \to \operatorname{Hom}_R(M,\, N \otimes_{R'} P). \tag{9.20.1}$$

Assume P is flat. If M is finitely generated, then θ is injective; if M is finitely presented, then θ is an isomorphism.

Proof: The map θ exists by Watts's Theorem, (**8.18**), with R' for R, applied to $\operatorname{Hom}_R(M, N \otimes_{R'} \bullet)$. Explicitly, $\theta(\varphi \otimes p)(m) = \varphi(m) \otimes p$.

Clearly, θ is bijective if $M = R$. So θ is bijective if $M = R^n$ for any n, as $\operatorname{Hom}_R(\bullet, Q)$ preserves finite direct sums for any Q by (**4.15**).

Assume that M is finitely generated. Then from (**5.20**), we obtain a presentation $R^{\oplus\Sigma} \to R^n \to M \to 0$, with Σ finite if P is finitely presented. Since θ is natural, it yields this commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(M, N) \otimes_{R'} P \arrow[r] \arrow[d, "\theta"] & \operatorname{Hom}_R(R^n, N) \otimes_{R'} P \arrow[r] \arrow[d, "\simeq"] & \operatorname{Hom}_R(R^{\oplus\Sigma}, N) \otimes_{R'} P \arrow[d] \\
0 \arrow[r] & \operatorname{Hom}_R(M, N \otimes_{R'} P) \arrow[r] & \operatorname{Hom}_R(R^n, N \otimes_{R'} P) \arrow[r] & \operatorname{Hom}_R(R^{\oplus\Sigma}, N \otimes_{R'} P)
\end{tikzcd}$$

Its rows are exact owing to the left exactness of Hom and to the flatness of P. The right-hand vertical map is bijective if Σ is finite. The assertion follows. □

Exercise (**9.21**). — Let R be a ring, R' an algebra, M and N modules. Show that there is a canonical map

$$\sigma\colon \operatorname{Hom}_R(M,\, N) \otimes_R R' \to \operatorname{Hom}_{R'}(M \otimes_R R',\, N \otimes_R R').$$

Assume R' is flat over R. Show that if M is finitely generated, then σ is injective, and that if M is finitely presented, then σ is an isomorphism.

Definition (**9.22**). — Let R be a ring, M a module. Let Λ_M be the category whose objects are the pairs (R^m, α) where $\alpha\colon R^m \to M$ is a homomorphism, and whose maps $(R^m, \alpha) \to (R^n, \beta)$ are the homomorphisms $\varphi\colon R^m \to R^n$ with $\beta\varphi = \alpha$.

Proposition (**9.23**). — *Let R be a ring, M a module, and $(R^m, \alpha) \mapsto R^m$ the forgetful functor from Λ_M to $((R\text{-mod}))$. Then $M = \varinjlim_{(R^m,\alpha)\in\Lambda_M} R^m$.*

Proof: By the UMP, the $\alpha\colon R^m \to M$ induce a map $\zeta\colon \varinjlim R^m \to M$. Let's show ζ is bijective. First, ζ is surjective, because each $x \in M$ is in the image of $(R,\, \alpha_x)$ where $\alpha_x(r) := rx$.

For injectivity, let $y \in \operatorname{Ker}(\zeta)$. By construction, $\bigoplus_{(R^m,\alpha)} R^m \to \varinjlim R^m$ is surjective; see the proof of (**6.10**). So y is in the image of some finite sum $\bigoplus_{(R^{m_i},\, \alpha_i)} R^{m_i}$. Set $m := \sum m_i$. Then $\bigoplus R^{m_i} = R^m$. Set $\alpha := \sum \alpha_i$. Then y is the image of some $y' \in R^m$ under the insertion $\iota_m\colon R^m \to \varinjlim R^m$. But $y \in \operatorname{Ker}(\zeta)$. So $\alpha(y') = 0$.

Let $\theta, \varphi\colon R \rightrightarrows R^m$ be the homomorphisms with $\theta(1) := y'$ and $\varphi(1) := 0$. They

13. Support

The spectrum of a ring is the following topological space: its points are the prime ideals, and each closed set consists of those primes containing a given ideal. The support of a module is the following subset: its points are the primes at which the localized module is nonzero. We relate the support to the closed set of the annihilator. We prove that a sequence is exact if and only if it is exact after localizing at every maximal ideal. We end this section by proving that the following conditions on a module ar equivalent: it is finitely generated and projective; it is finitely presented and flat; and it is locally free of finite rank.

(13.1) (*Spectrum of a ring*). — Let R be a ring. Its set of prime ideals is denoted $\operatorname{Spec}(R)$, and is called the (prime) **spectrum** of R.

Let $\mathfrak{a}$ be an ideal. Let $\mathbf{V}(\mathfrak{a})$ denote the subset of $\operatorname{Spec}(R)$ consisting of those primes that contain $\mathfrak{a}$. We call $\mathbf{V}(\mathfrak{a})$ the **variety** of $\mathfrak{a}$.

Let $\mathfrak{b}$ be a second ideal. Obviously, if $\mathfrak{a} \subset \mathfrak{b}$, then $\mathbf{V}(\mathfrak{b}) \subset \mathbf{V}(\mathfrak{a})$. Conversely, if $\mathbf{V}(\mathfrak{b}) \subset \mathbf{V}(\mathfrak{a})$, then $\mathfrak{a} \subset \sqrt{\mathfrak{b}}$, owing to the Scheinnullstellensatz **(3.29)**. Therefore, $\mathbf{V}(\mathfrak{a}) = \mathbf{V}(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$. Further, **(2.2)** yields

$$\mathbf{V}(\mathfrak{a}) \cup \mathbf{V}(\mathfrak{b}) = \mathbf{V}(\mathfrak{a} \cap \mathfrak{b}) = \mathbf{V}(\mathfrak{a}\mathfrak{b}).$$

A prime ideal $\mathfrak{p}$ contains the ideals $\mathfrak{a}_\lambda$ in an arbitrary collection if and only if $\mathfrak{p}$ contains their sum $\sum \mathfrak{a}_\lambda$; hence,

$$\bigcap \mathbf{V}(\mathfrak{a}_\lambda) = \mathbf{V}\left(\sum \mathfrak{a}_\lambda\right).$$

Finally, $\mathbf{V}(R) = \emptyset$, and $\mathbf{V}(\langle 0 \rangle) = \operatorname{Spec}(R)$. Thus the subsets $\mathbf{V}(\mathfrak{a})$ of $\operatorname{Spec}(R)$ are the closed sets of a topology; it is called the **Zariski topology**.

Given an element $f \in R$, we call the open set

$$\mathbf{D}(f) := \operatorname{Spec}(R) - \mathbf{V}(\langle f \rangle)$$

a **principal open set**. These sets form a basis for the topology of $\operatorname{Spec}(R)$; indeed, given any prime $\mathfrak{p} \not\supset \mathfrak{a}$, there is an $f \in \mathfrak{a} - \mathfrak{p}$, and so $\mathfrak{p} \in \mathbf{D}(f) \subset \operatorname{Spec}(R) - \mathbf{V}(\mathfrak{a})$. Further, $f, g \notin \mathfrak{p}$ if and only if $fg \notin \mathfrak{p}$, for any $f, g \in R$ and prime $\mathfrak{p}$; in other words,

$$\mathbf{D}(f) \cap \mathbf{D}(g) = \mathbf{D}(fg). \qquad \textbf{(13.1.1)}$$

A ring map $\varphi\colon R \to R'$ induces a set map

$$\operatorname{Spec}(\varphi)\colon \operatorname{Spec}(R') \to \operatorname{Spec}(R) \quad \text{by} \quad \operatorname{Spec}(\varphi)(\mathfrak{p}') := \varphi^{-1}(\mathfrak{p}'). \qquad \textbf{(13.1.2)}$$

Notice $\varphi^{-1}(\mathfrak{p}') \supset \mathfrak{a}$ if and only if $\mathfrak{p}' \supset \mathfrak{a}R'$; so $\operatorname{Spec}(\varphi)^{-1}\,\mathbf{V}(\mathfrak{a}) = \mathbf{V}(\mathfrak{a}R')$. Hence $\operatorname{Spec}(\varphi)$ is continuous. Thus $\operatorname{Spec}(\bullet)$ is a contravariant functor from ((Rings)) to ((Top spaces)).

For example, the quotient map $R \to R/\mathfrak{a}$ induces a topological embedding

$$\operatorname{Spec}(R/\mathfrak{a}) \hookrightarrow \operatorname{Spec}(R), \qquad \textbf{(13.1.3)}$$

whose image is $\mathbf{V}(\mathfrak{a})$, owing to **(1.9)** and **(2.8)**. Furthermore, the localization map $R \to R_f$ induces a topological embedding

$$\operatorname{Spec}(R_f) \hookrightarrow \operatorname{Spec}(R), \qquad \textbf{(13.1.4)}$$

whose image is $\mathbf{D}(f)$, owing to **(11.20)**.

Exercise **(13.13)**. — Let $\varphi\colon R \to R'$ be a ring map, and $\mathfrak{b}$ an ideal of R'. Set $\varphi^* := \operatorname{Spec}(\varphi)$. Show (1) that the closure $\overline{\varphi^*(\mathbf{V}(\mathfrak{b}))}$ in $\operatorname{Spec}(R)$ is equal to $\mathbf{V}(\varphi^{-1}\mathfrak{b})$ and (2) that $\varphi^*(\operatorname{Spec}(R'))$ is dense in $\operatorname{Spec}(R)$ if and only if $\operatorname{Ker}(\varphi) \subset \operatorname{nil}(R)$.

Exercise **(13.14)**. — Let R be a ring, R' a flat algebra with structure map φ. Show that R' is faithfully flat if and only if $\operatorname{Spec}(\varphi)$ is surjective.

Exercise **(13.15)**. — Let $\varphi\colon R \to R'$ be a flat map of rings, $\mathfrak{q}$ a prime of R', and $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. Show that the induced map $\operatorname{Spec}(R'_{\mathfrak{q}}) \to \operatorname{Spec}(R_{\mathfrak{p}})$ is surjective.

Exercise **(13.16)**. — Let R be a ring. Given $f \in R$, set $S_f := \{f^n \mid n \geq 0\}$, and let $\overline{S}_f$ denote its saturation; see **(3.17)**. Given $f, g \in R$, show that the following conditions are equivalent:

(1) $\mathbf{D}(g) \subset \mathbf{D}(f)$. (2) $\mathbf{V}(\langle g\rangle) \supset \mathbf{V}(\langle f\rangle)$. (3) $\sqrt{\langle g\rangle} \subset \sqrt{\langle f\rangle}$.
(4) $\overline{S}_f \subset \overline{S}_g$. (5) $g \in \sqrt{\langle f\rangle}$. (6) $f \in \overline{S}_g$.
(7) there is a unique R-algebra map $\varphi^f_g\colon \overline{S}_f^{-1}R \to \overline{S}_g^{-1}R$.
(8) there is an R-algebra map $R_f \to R_g$.

Show that, if these conditions hold, then the map in (8) is equal to φ^f_g.

Exercise **(13.17)**. — Let R be a ring. (1) Show that $\mathbf{D}(f) \mapsto R_f$ is a well-defined contravariant functor from the category of principal open sets and inclusions to $((R\text{-alg}))$. (2) Given $\mathfrak{p} \in \operatorname{Spec}(R)$, show $\varinjlim_{\mathbf{D}(f)\ni\mathfrak{p}} R_f = R_{\mathfrak{p}}$.

Exercise **(13.18)**. — A topological space is called **irreducible** if it's nonempty and if every pair of nonempty open subsets meet. Let R be a ring. Set $X := \operatorname{Spec}(R)$ and $\mathfrak{n} := \operatorname{nil}(R)$. Show that X is irreducible if and only if $\mathfrak{n}$ is prime.

Exercise **(13.19)**. — Let X be a topological space, Y an irreducible subspace.
(1) Show that the closure $\overline{Y}$ of Y is also irreducible.
(2) Show that Y is contained in a maximal irreducible subspace.
(3) Show that the maximal irreducible subspaces of X are closed, and cover X. They are called its **irreducible components**. What are they if X is Hausdorff?
(4) Let R be a ring, and take $X := \operatorname{Spec}(R)$. Show that its irreducible components are the closed sets $\mathbf{V}(\mathfrak{p})$ where $\mathfrak{p}$ is a minimal prime.

Proposition **(13.20)**. — *Let R be a ring, $X := \operatorname{Spec}(R)$. Then X is **quasi-compact:** if $X = \bigcup_{\lambda\in\Lambda} U_\lambda$ with U_λ open, then $X = \bigcup_{i=1}^n U_{\lambda_i}$ for some $\lambda_i \in \Lambda$.*

Proof: Say $U_\lambda = X - \mathbf{V}(\mathfrak{a}_\lambda)$. As $X = \bigcup_{\lambda\in\Lambda} U_\lambda$, then $\emptyset = \bigcap \mathbf{V}(\mathfrak{a}_\lambda) = \mathbf{V}\big(\sum \mathfrak{a}_\lambda\big)$. So $\sum \mathfrak{a}_\lambda$ lies in no prime ideal. Hence there are $\lambda_1, \dots, \lambda_n \in \Lambda$ and $f_{\lambda_i} \in \mathfrak{a}_{\lambda_i}$ with $1 = \sum f_{\lambda_i}$. So $R = \sum \mathfrak{a}_{\lambda_i}$. So $\emptyset = \bigcap \mathbf{V}(\mathfrak{a}_{\lambda_i}) = \mathbf{V}\big(\sum \mathfrak{a}_{\lambda_i}\big)$. Thus $X = \bigcup U_{\lambda_i}$. □

Exercise **(13.21)**. — Let R be a ring, $X := \operatorname{Spec}(R)$, and U an open subset. Show U is quasi-compact if and only if $X - U = \mathbf{V}(\mathfrak{a})$ where $\mathfrak{a}$ is finitely generated.

Exercise **(13.22)**. — Let R be a ring, M a module, $m \in M$. Set $X := \operatorname{Spec}(R)$. Assume $X = \bigcup \mathbf{D}(f_\lambda)$ for some f_λ, and $m/1 = 0$ in M_{f_λ} for all λ. Show $m = 0$.

Exercise **(13.23)**. — Let R be a ring; set $X := \operatorname{Spec}(R)$. Prove that the four following conditions are equivalent:

(1) $R/\operatorname{nil}(R)$ is absolutely flat.
(2) X is Hausdorff.

$M_\mathfrak{m}$. Thus (**13.44**) yields (1).

Assume M is locally finitely presented. Then M is finitely generated by (1). So there is a surjection $R^k \twoheadrightarrow M$. Let K be its kernel. Then K is locally finitely generated owing to (**5.26**). Hence K too is finitely generated by (1). So there is a surjection $R^\ell \twoheadrightarrow K$. It yields the desired finite presentation $R^\ell \to R^k \to M \to 0$. Thus (2) holds. □

Theorem (**13.51**). — *These conditions on an R-module P are equivalent:*

(1) *P is finitely generated and projective.*
(2) *P is finitely presented and flat.*
(3) *P is finitely presented, and $P_\mathfrak{m}$ is free over $R_\mathfrak{m}$ at each maximal ideal $\mathfrak{m}$.*
(4) *P is locally free of finite rank.*
(5) *P is finitely generated, and for each $\mathfrak{p} \in \mathrm{Spec}(R)$, there are f and n such that $\mathfrak{p} \in \boldsymbol{D}(f)$ and $P_\mathfrak{q}$ is free of rank n over $R_\mathfrak{q}$ at each $\mathfrak{q} \in \boldsymbol{D}(f)$.*

Proof: Condition (1) implies (2) by (**10.20**).

Let $\mathfrak{m}$ be a maximal ideal. Then $R_\mathfrak{m}$ is local by (**11.22**). If P is finitely presented, then $P_\mathfrak{m}$ is finitely presented, because localization preserves direct sums and cokernels by (**12.11**).

Assume (2). Then $P_\mathfrak{m}$ is flat by (**13.46**), so free by (**10.20**). Thus (3) holds.

Assume (3). Fix a surjective map $\alpha\colon M \to N$. Then $\alpha_\mathfrak{m}\colon M_\mathfrak{m} \to N_\mathfrak{m}$ is surjective. So $\mathrm{Hom}(P_\mathfrak{m}, \alpha_\mathfrak{m})\colon \mathrm{Hom}(P_\mathfrak{m}, M_\mathfrak{m}) \to \mathrm{Hom}(P_\mathfrak{m}, N_\mathfrak{m})$ is surjective by (**5.23**) and (**5.22**). But $\mathrm{Hom}(P_\mathfrak{m}, \alpha_\mathfrak{m}) = \mathrm{Hom}(P, \alpha)_\mathfrak{m}$ by (**12.25**) as P is finitely presented. Further, $\mathfrak{m}$ is arbitrary. Hence $\mathrm{Hom}(P, \alpha)$ is surjective by (**13.43**). Therefore, P is projective by (**5.23**). Thus (1) holds.

Again assume (3). Given any prime $\mathfrak{p}$, take a maximal ideal $\mathfrak{m}$ containing it. By hypothesis, $P_\mathfrak{m}$ is free; its rank is finite as $P_\mathfrak{m}$ is finitely generated. By (**12.24**)(2), there is $f \in R - \mathfrak{m}$ such that P_f is free of finite rank over R_f. Thus (4) holds.

Assume (4). Then P is locally finitely presented. So P is finitely presented by (**13.50**)(2). Further, given $\mathfrak{p} \in \mathrm{Spec}(R)$, there are $f \in R - \mathfrak{p}$ and n such that P_f is free of rank n over R_f. Given $\mathfrak{q} \in \mathbf{D}(f)$, let S be the image of $R - \mathfrak{q}$ in R_f. Then (**12.5**) yields $P_\mathfrak{q} = S^{-1}(P_f)$. Hence $P_\mathfrak{q}$ is free of rank n over $R_\mathfrak{q}$. Thus (5) holds. Further, (3) results from taking $\mathfrak{p} := \mathfrak{m}$ and $\mathfrak{q} := \mathfrak{m}$.

Finally, assume (5), and let's prove (4). Given $\mathfrak{p} \in \mathrm{Spec}(R)$, let f and n be provided by (5). Take a free basis $p_1/f^{k_1}, \dots, p_n/f^{k_n}$ of $P_\mathfrak{p}$ over $R_\mathfrak{p}$. The p_i define a map $\alpha\colon R^n \to P$, and $\alpha_\mathfrak{p}\colon R^n_\mathfrak{p} \to P_\mathfrak{p}$ is bijective, in particular, surjective.

As P is finitely generated, (**12.24**)(1) provides $g \in R - \mathfrak{p}$ such that $\alpha_g\colon R^n_g \to P_g$ is surjective. It follows that $\alpha_\mathfrak{q}\colon R^n_\mathfrak{q} \to P_\mathfrak{q}$ is surjective for every $\mathfrak{q} \in \mathbf{D}(g)$. If also $\mathfrak{q} \in \mathbf{D}(f)$, then by hypothesis $P_\mathfrak{q} \simeq R^n_\mathfrak{q}$. So $\alpha_\mathfrak{q}$ is bijective by (**10.4**).

Set $h := fg$. Clearly, $\mathbf{D}(f) \cap \mathbf{D}(g) = \mathbf{D}(h)$. By (**13.1**), $\mathbf{D}(h) = \mathrm{Spec}(R_h)$. Clearly, $\alpha_\mathfrak{q} = (\alpha_h)_{(\mathfrak{q}R_h)}$ for all $\mathfrak{q} \in \mathbf{D}(h)$. Hence $\alpha_h\colon R^n_h \to P_h$ is bijective owing to (**13.43**) with R_h for R. Thus (4) holds. □

Exercise (**13.52**). — Given n, prove an R-module P is locally free of rank n if and only if P is finitely generated and $P_\mathfrak{m} \simeq R^n_\mathfrak{m}$ holds at each maximal ideal $\mathfrak{m}$.

Exercise (**13.53**). — Let A be a semilocal ring, P a locally free module of rank n. Show that P is free of rank n.

Exercise (**13.54**). — Let R be a ring, M a finitely presented module, $n \geq 0$. Show that M is locally free of rank n if and only if $F_{n-1}(M) = \langle 0 \rangle$ and $F_n(M) = R$.

Exercise (**14.14**). — Let A be a reduced local ring with residue field k and finite set Σ of minimal primes. For each $\mathfrak{p} \in \Sigma$, set $K(\mathfrak{p}) := \mathrm{Frac}(A/\mathfrak{p})$. Let P be a finitely generated module. Show that P is free of rank r if and only if $\dim_k(P \otimes_A k) = r$ and $\dim_{K(\mathfrak{p})}(P \otimes_A K(\mathfrak{p})) = r$ for each $\mathfrak{p} \in \Sigma$.

Exercise (**14.15**). — Let A be a reduced local ring with residue field k and a finite set of minimal primes. Let P be a finitely generated module, B an A-algebra with $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ surjective. Show that P is a free A-module of rank r if and only if $P \otimes B$ is a free B-module of rank r.

(14.16) (*Arbitrary normal rings*). — An arbitrary ring R is said to be **normal** if $R_\mathfrak{p}$ is a normal domain for every prime $\mathfrak{p}$. If R is a domain, then this definition recovers that in **(10.30)**, owing to **(11.32)**.

Exercise (**14.17**). — Let R be a ring, $\mathfrak{p}_1 \ldots, \mathfrak{p}_r$ all its minimal primes, and K the total quotient ring. Prove that these three conditions are equivalent:

(1) R is normal.
(2) R is reduced and integrally closed in K.
(3) R is a finite product of normal domains R_i.

Assume the conditions hold. Prove the R_i are equal to the $R/\mathfrak{p}_j$ in some order.

Finally, $p_{\mathfrak{q}}(M, n) - p(F^\bullet M, n)$ is a polynomial with degree at most $d-1$ and positive leading coefficient; also, d and $e(1)$ are the same for every stable $\mathfrak{q}$-filtration.

Proof: The proof of **(20.13)** shows that $G^\bullet R'$ and $G^\bullet M$ satisfy the hypotheses of **(20.8)**. So **(20.8.1)** and **(20.13.1)** yield **(20.14.1)**. In turn, **(20.13.1)** yields **(20.14.2)** by the argument in the second paragraph of the proof of **(20.8)**.

Finally, as $F^\bullet M$ is a stable $\mathfrak{q}$-filtration, there is an m such that

$$F^n M \supset \mathfrak{q}^n M \supset \mathfrak{q}^n F^m M = F^{n+m} M$$

for all $n \geq 0$. Dividing into M and extracting lengths, we get

$$\ell(M/F^n M) \leq \ell(M/\mathfrak{q}^n M) \leq \ell(M/F^{n+m} M).$$

Therefore, **(20.14.2)** yields

$$p(F^\bullet M, n) \leq p_{\mathfrak{q}}(M, n) \leq p(F^\bullet M, n+m) \quad \text{for } n \gg 0.$$

The two extremes are polynomials in n with the same degree d and the same leading coefficient c where $c := e(1)/d!$. Dividing by n^d and letting $n \to \infty$, we conclude that the polynomial $p_{\mathfrak{q}}(M, n)$ also has degree d and leading coefficient c.

Thus the degree and leading coefficient are the same for every stable $\mathfrak{q}$-filtration. Also $p_{\mathfrak{q}}(M, n) - p(F^\bullet M, n)$ has degree at most $d-1$ and positive leading coefficient, owing to cancellation of the two leading terms and to the first inequality. □

Exercise **(20.15)**. — Let R be a Noetherian ring, $\mathfrak{q}$ an ideal, and M a finitely generated module. Assume $\ell(M/\mathfrak{q}M) < \infty$. Set $\mathfrak{m} := \sqrt{\mathfrak{q}}$. Show

$$\deg p_{\mathfrak{m}}(M, n) = \deg p_{\mathfrak{q}}(M, n).$$

(20.16) (*Rees Algebras*). — Let R be an arbitrary ring, $\mathfrak{q}$ an ideal. The sum

$$\mathcal{R}(\mathfrak{q}) := \bigoplus_{n \in \mathbb{Z}} \mathcal{R}_n(\mathfrak{q}) \quad \text{with } \mathcal{R}_n(\mathfrak{q}) := \begin{cases} R & \text{if } n \leq 0, \\ \mathfrak{q}^n & \text{if } n > 0 \end{cases}$$

is canonically an R-algebra, known as the **extended Rees Algebra** of $\mathfrak{q}$.

Let M be a module with a $\mathfrak{q}$-filtration $F^\bullet M$. Then the sum

$$\mathcal{R}(F^\bullet M) := \bigoplus_{n \in \mathbb{Z}} F^n M$$

is canonically an $\mathcal{R}(\mathfrak{q})$-module, known as the **Rees Module** of $F^\bullet M$.

Lemma **(20.17)**. — *Let R be a Noetherian ring, $\mathfrak{q}$ an ideal, M a finitely generated module with a $\mathfrak{q}$-filtration $F^\bullet M$. Then $\mathcal{R}(\mathfrak{q})$ is algebra finite over R. Also, $F^\bullet M$ is stable if and only if $\mathcal{R}(F^\bullet M)$ is module finite over $\mathcal{R}(\mathfrak{q})$ and $\bigcup F^n M = M$.*

Proof: As R is Noetherian, $\mathfrak{q}$ is finitely generated, say by $x_1, \ldots, x_r$. View the x_i as in $\mathcal{R}_1(\mathfrak{q})$ and $1 \in R$ as in $\mathcal{R}_{-1}(\mathfrak{q})$. These $r+1$ elements generate $\mathcal{R}(\mathfrak{q})$ over R.

Suppose that $F^\bullet M$ is stable: say $F^\mu M = M$ and $\mathfrak{q}^n F^\nu M F^{n+\nu} M$ for $n > 0$. Then $\bigcup F^n M = M$. Further, $\mathcal{R}(F^\bullet M)$ is generated by $F^\mu M, \ldots, F^\nu M$ over $\mathcal{R}(\mathfrak{q})$. But R is Noetherian and M is finitely generated over R; hence, every $F^n M$ is finitely generated over R. Thus $\mathcal{R}(F^\bullet M)$ is a finitely generated $\mathcal{R}(\mathfrak{q})$-module.

Conversely, suppose that $\mathcal{R}(F^\bullet M)$ is generated over $\mathcal{R}(\mathfrak{q})$ by $m_1, \ldots, m_s$. Say $m_i = \sum_{j=\mu}^{\nu} m_{ij}$ with $m_{ij} \in F^j M$ for some uniform $\mu \leq \nu$. Then given n, any $m \in F^n M$ can be written as $m = \sum f_{ij} m_{ij}$ with $f_{ij} \in \mathcal{R}_{n-j}(\mathfrak{q})$. Hence if $n \leq \mu$, then $F^n M \subset F^\mu M$. Suppose $\bigcup F^n M = M$. Then $F^\mu M = M$. But if $j \leq \nu \leq n$, then $f_{ij} \in \mathfrak{q}^{n-j} = \mathfrak{q}^{n-\nu}\mathfrak{q}^{\nu-j}$. Thus $\mathfrak{q}^{n-\nu} F^\nu M = F^n M$. Thus $F^\bullet M$ is stable. □

Lemma **(20.18)** (Artin–Rees). — *Let R be a Noetherian ring, M a finitely generated module, N a submodule, $\mathfrak{q}$ an ideal, $F^\bullet M$ a stable $\mathfrak{q}$-filtration. Set*

$$F^nN := N \cap F^nM \quad \textit{for } n \in \mathbb{Z}.$$

Then the F^nN form a stable $\mathfrak{q}$-filtration $F^\bullet N$.

Proof: By **(20.17)**, the extended Rees Algebra $\mathcal{R}(\mathfrak{q})$ is finitely generated over R, so Noetherian by the Hilbert Basis Theorem **(16.12)**. By **(20.17)**, the module $\mathcal{R}(F^\bullet M)$ is finitely generated over $\mathcal{R}(\mathfrak{q})$, so Noetherian by **(16.19)**. Clearly, $F^\bullet N$ is a $\mathfrak{q}$-filtration; hence, $\mathcal{R}(F^\bullet N)$ is a submodule of $\mathcal{R}(F^\bullet M)$, so finitely generated. But $\bigcup F^nM = M$, so $\bigcup F^nN = N$. Thus $F^\bullet N$ is stable by **(20.17)**. □

Exercise **(20.19)**. — Derive the Krull Intersection Theorem, **(18.29)**, from the Artin–Rees Lemma, **(20.18)**.

Proposition **(20.20)**. — *Let R be a Noetherian ring, $\mathfrak{q}$ an ideal, and*

$$0 \to M' \to M \to M'' \to 0$$

an exact sequence of finitely generated modules. Then $M/\mathfrak{q}M$ has finite length if and only if $M'/\mathfrak{q}M'$ and $M''/\mathfrak{q}M''$ do. If so, then the polynomial

$$p_{\mathfrak{q}}(M',n) - p_{\mathfrak{q}}(M,n) + p_{\mathfrak{q}}(M'',n)$$

has degree at most $\deg p_{\mathfrak{q}}(M',n) - 1$ and has positive leading coefficient; also then

$$\deg p_{\mathfrak{q}}(M,n) = \max\{\deg p_{\mathfrak{q}}(M',n),\ \deg p_{\mathfrak{q}}(M'',n)\}.$$

Proof: First off, **(13.31)** and **(13.27)**(1) and **(13.31)** again yield

$$\begin{aligned}\operatorname{Supp}(M/\mathfrak{q}M) &= \operatorname{Supp}(M) \bigcap \mathbf{V}(\mathfrak{q}) = \big(\operatorname{Supp}(M') \bigcup \operatorname{Supp}(M'')\big) \bigcap \mathbf{V}(\mathfrak{q})\\ &= \big(\operatorname{Supp}(M') \bigcap \mathbf{V}(\mathfrak{q})\big) \bigcup \big(\operatorname{Supp}(M'') \bigcap \mathbf{V}(\mathfrak{q})\big)\\ &= \operatorname{Supp}(M'/\mathfrak{q}M') \bigcup \operatorname{Supp}(M''/\mathfrak{q}M'').\end{aligned}$$

Hence $M/\mathfrak{q}M$ has finite length if and only if $M'/\mathfrak{q}M'$ and $M''/\mathfrak{q}M''$ do by **(19.4)**.

For $n \in \mathbb{Z}$, set $F^nM' := M' \bigcap \mathfrak{q}^nM$. Then the F^nM' form a stable $\mathfrak{q}$-filtration $F^\bullet M'$ by the Artin–Rees Lemma. Form this canonical commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & F^nM' & \to & \mathfrak{q}^nM & \to & \mathfrak{q}^nM'' & \to & 0\\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

Its rows are exact. So the Nine Lemma yields this exact sequence:

$$0 \to M'/F^nM' \to M/\mathfrak{q}^nM \to M''/\mathfrak{q}^nM'' \to 0.$$

Assume $M/\mathfrak{q}M$ has finite length. Then Additivity of Length and **(20.14)** yield

$$p(F^\bullet M',\, n) - p_{\mathfrak{q}}(M,\, n) + p_{\mathfrak{q}}(M'',\, n) = 0. \qquad \textbf{(20.20.1)}$$

Hence $p_{\mathfrak{q}}(M',\, n) - p_{\mathfrak{q}}(M,\, n) + p_{\mathfrak{q}}(M'',\, n)$ is equal to $p_{\mathfrak{q}}(M',\, n) - p(F^\bullet M',\, n)$. But by **(20.14)** again, the latter is a polynomial with degree at most $\deg p_{\mathfrak{q}}(M',\, n) - 1$ and positive leading coefficient.

Finally, $\deg p_{\mathfrak{q}}(M,n) = \max\{\deg p(M'_\bullet, n),\ \deg p_{\mathfrak{q}}(M'', n)\}$ owing to **(20.20.1)**, as the leading coefficients of $p(M'_\bullet, n)$ and $p_{\mathfrak{q}}(M'', n)$ are both positive, so cannot cancel. But $\deg p(M'_\bullet, n) = \deg p_{\mathfrak{q}}(M', n)$ by **(20.14)**, completing the proof. □

Theorem **(20.27)**. — *Let R be a Noetherian graded ring, M a finitely generated graded module, N a homogeneous submodule. Then all the associated primes of M/N are homogeneous, and N admits an irredundant primary decomposition in which all the primary submodules are homogeneous.*

Proof: Let $N = \bigcap Q_j$ be any primary decomposition; one exists by **(18.21)**. Let $Q_j^* \subset Q_j$ be the submodule generated by the homogeneous elements of Q_j. Trivially, $\bigcap Q_j^* \subset \bigcap Q_j = N \subset \bigcap Q_j^*$. Further, each Q_j^* is clearly homogeneous, and is primary by **(20.26)**. Thus $N = \bigcap Q_j^*$ is a primary decomposition into homogeneous primary submodules. And, owing to **(18.19)**, it is irredundant if $N = \bigcap Q_j$ is, as both decompositions have minimal length. Finally, M/Q_j^* is graded by **(20.21)**; so each associated prime is homogeneous by **(18.20)** and **(20.25)**. □

(20.28) (*Graded Domains*). — Let $R = \bigoplus_{n\geq 0} R_n$ be a graded domain, and set $K := \mathrm{Frac}(R)$. We call $z \in K$ **homogeneous of degree** $n \in \mathbb{Z}$ if $z = x/y$ with $x \in R_m$ and $y \in R_{m-n}$. Clearly, n is well defined.

Let K_n be the set of all such z, plus 0. Then $K_m K_n \subset K_{m+n}$. Clearly, the canonical map $\bigoplus_{n\in\mathbb{Z}} K_n \to K$ is injective. Thus $\bigoplus_{n\geq 0} K_n$ is a graded subring of K. Further, K_0 is a field.

The n with $K_n \neq 0$ form a subgroup of $\mathbb{Z}$. So by renumbering, we may assume $K_1 \neq 0$. Fix any nonzero $x \in K_1$. Clearly, x is transcendental over K_0. If $z \in K_n$, then $z/x^n \in K_0$. Hence $R \subset K_0[x]$. So **(2.3)** yields $K = K_0(x)$.

Any $w \in \bigoplus K_n$ can be written $w = a/b$ with $a, b \in R$ and b homogeneous: say $w = \sum(a_n/b_n)$ with $a_n, b_n \in R$ homogeneous; set $b := \prod b_n$ and $a := \sum(a_n b/b_n)$.

Theorem **(20.29)**. — *Let R be a Noetherian graded domain, $K := \mathrm{Frac}(R)$, and $\overline{R}$ the integral closure of R in K. Then $\overline{R}$ is a graded subring of K.*

Proof: Use the setup of **(20.28)**. Since $K_0[x]$ is a polynomial ring over a field, it is normal by **(10.34)**. Hence $\overline{R} \subset K_0[x]$. So every $y \in \overline{R}$ can be written as $y = \sum_{i=r}^{r+n} y_i$, with y_i homogeneous and nonzero. Let's show $y_i \in \overline{R}$ for all i.

Since y is integral over R, the R-algebra $R[y]$ is module finite by **(10.23)**. So **(20.28)** yields a homogeneous $b \in R$ with $bR[y] \subset R$. Hence $by^j \in R$ for all $j \geq 0$. But R is graded. Hence $by_r^j \in R$. Set $z := 1/b$. Then $y_r^j \in Rz$. Since R is Noetherian, the R-algebra $R[y_r]$ is module finite. Hence $y_r \in \overline{R}$. Then $y - y_r \in \overline{R}$. Thus $y_i \in \overline{R}$ for all i by induction on n. Thus $\overline{R}$ is graded. □

Exercise **(20.30)**. — Under the conditions of **(20.8)**, assume that R is a domain and that its integral closure $\overline{R}$ in $\mathrm{Frac}(R)$ is a finitely generated R-module.

(1) Prove that there is a homogeneous $f \in R$ with $R_f = \overline{R}_f$.

(2) Prove that the Hilbert Polynomials of R and $\overline{R}$ have the same degree and same leading coefficient.

$$m_n - m_{n'} = (m_n - m_{n+1}) + (m_{n+1} - m_{n+2}) + \cdots + (m_{n'-1} - m_{n'}).$$

An $m \in M$ is called a **limit** of (m_n) if, given n_0, there's n_1 with $m - m_n \in F^{n_0}M$ for all $n \ge n_1$. If every Cauchy sequence has a limit, then M is called **complete.**

The Cauchy sequences form a module under termwise addition and scalar multiplication. The sequences with 0 as a limit form a submodule. The quotient module is denoted $\widehat{M}$ and called the (separated) **completion**. There is a canonical homomorphism, which carries $m \in M$ to the class of the constant sequence (m):

$$\kappa\colon M \to \widehat{M} \quad \text{by} \quad \kappa m := (m).$$

If M is complete, but not separated, then κ is surjective, but not bijective.

It is easy to check that the notions of Cauchy sequence and limit depend only on the topology. Further, $\widehat{M}$ is separated and complete with respect to the filtration $F^k\widehat{M} := (F^kM)\widehat{}$ where $(F^kM)\widehat{}$ is the completion of F^kM arising from the intersections $F^kM \cap F^nM$ for all n. In addition, κ *is the universal continuous R-linear map from M into a separated and complete, filtered $\widehat{R}$-module.*

Again, let $\mathfrak{a}$ be an ideal. Under termwise multiplication of Cauchy sequences, $\widehat{R}$ *is a ring,* $\kappa\colon R \to \widehat{R}$ *is a ring homomorphism, and* $\widehat{M}$ *is an* $\widehat{R}$*-module.* Further, $M \mapsto \widehat{M}$ *is a linear functor* from $((R\text{-mod}))$ to $((\widehat{R}\text{-mod}))$.

For example, let R' be a ring, and $R := R'[X_1, \ldots, X_r]$ the polynomial ring in r variables. Set $\mathfrak{a} := \langle X_1, \ldots, X_r\rangle$. Then a sequence $(m_n)_{n\ge0}$ of polynomials is Cauchy if and only if, given n_0, there's n_1 such that, for all $n \ge n_1$, the m_n agree in degree less than n_0. Thus $\widehat{R}$ is just the power series ring $R'[[X_1, \ldots, X_r]]$.

For another example, take a prime integer p, and set $\mathfrak{a} := \langle p\rangle$. Then a sequence $(m_n)_{n\ge0}$ of integers is Cauchy if and only if, given n_0, there's n_1 such that, for all $n, n' \ge n_1$, the difference $m_{n'} - m_n$ is a multiple of p^{n_0}. The completion of $\mathbb{Z}$ is called the p**-adic integers**, and consists of the sums $\sum_{i=0}^\infty z_ip^i$ with $0 \le z_i < p$.

Proposition **(22.2)**. — *Let R be a ring, and $\mathfrak{a}$ an ideal. Then $\widehat{\mathfrak{a}} \subset \operatorname{rad}(\widehat{R})$.*

Proof: Recall from **(22.1)** that $\widehat{R}$ is complete in the $\widehat{\mathfrak{a}}$-adic topology. Hence for $x \in \widehat{\mathfrak{a}}$, we have $1/(1-x) = 1 + x + x^2 + \cdots$ in $\widehat{R}$. Thus $\widehat{\mathfrak{a}} \subset \operatorname{rad}(\widehat{R})$ by **(3.2)**. □

Exercise **(22.3)**. — In the 2-adic integers, evaluate the sum $1 + 2 + 4 + 8 + \cdots$.

Exercise **(22.4)**. — Let R be a ring, $\mathfrak{a}$ an ideal, and M a module. Prove that the following three conditions are equivalent:

(1) $\kappa\colon M \to \widehat{M}$ is injective; (2) $\bigcap \mathfrak{a}^nM = \langle 0\rangle$; (3) M is separated.

Assume R is Noetherian and M finitely generated. Assume either (a) $\mathfrak{a} \subset \operatorname{rad}(R)$ or (b) R is a domain, $\mathfrak{a}$ is proper, and M is torsionfree. Conclude $M \subset \widehat{M}$.

(22.5) (*Inverse limits*). — Let R be a ring. Given R-modules Q_n equipped with linear maps $\alpha_n^{n+1}\colon Q_{n+1} \to Q_n$ for n, their **inverse limit** $\varprojlim Q_n$ is the submodule of $\prod Q_n$ of all vectors (q_n) with $\alpha_n^{n+1}q_{n+1} = q_n$ for all n.

Given Q_n and α_n^{n+1} for all $n \in \mathbb{Z}$, use only those for n in the present context.

Define $\theta\colon \prod Q_n \to \prod Q_n$ by $\theta(q_n) := (q_n - \alpha_n^{n+1}q_{n+1})$. Then

$$\varprojlim Q_n = \operatorname{Ker}\theta. \quad \text{Set } \varprojlim{}^1 Q_n := \operatorname{Coker}\theta. \qquad \textbf{(22.5.1)}$$

Plainly, $\varprojlim Q_n$ *has this UMP: given maps $\beta_n\colon P \to Q_n$ with $\alpha_n^{n+1}\beta_{n+1} = \beta_n$, there's a unique map $\beta\colon P \to \varprojlim Q_n$ with $\pi_n\beta = \beta_n$ for all n.*

Further, the UMP yields the following natural R-linear isomorphism:

$$\varprojlim \operatorname{Hom}(P,\, Q_n) = \operatorname{Hom}(P,\, \varprojlim Q_n).$$

(The notion of inverse limit is formally dual to that of direct limit.)

For example, let R' be a ring, and $R := R'[X_1, \dots, X_r]$ the polynomial ring in r variables. Set $\mathfrak{m} := \langle X_1, \dots, X_r \rangle$ and $R_n := R/\mathfrak{m}^{n+1}$. Then R_n is just the R-algebra of polynomials of degree at most n, and the canonical map $\alpha_n^{n+1}\colon R_{n+1} \to R_n$ is just truncation. Thus $\varprojlim R_n$ is equal to the power series ring $R'[[X_1, \dots, X_r]]$.

For another example, take a prime integer p, and set $\mathbb{Z}_n := \mathbb{Z}/\langle p^{n+1} \rangle$. Then $\mathbb{Z}_n$ is just the ring of sums $\sum_{i=0}^{n} z_i p^i$ with $0 \le z_i < p$, and the canonical map $\alpha_n^{n+1}\colon \mathbb{Z}_{n+1} \to \mathbb{Z}_n$ is just truncation. Thus $\varprojlim \mathbb{Z}_n$ is just the ring of p-adic integers.

Exercise (**22.6**). — Let R be a ring. Given R-modules Q_n equipped with linear maps $\alpha_n^{n+1}\colon Q_{n+1} \to Q_n$ for $n \ge 0$, set $\alpha_n^m := \alpha_n^{n+1} \cdots \alpha_{m-1}^m$ for $m > n$. We say the Q_n satisfy the **Mittag-Leffler Condition** if the descending chain

$$Q_n \supset \alpha_n^{n+1} Q_{n+1} \supset \alpha_n^{n+2} Q_{n+2} \supset \cdots \supset \alpha_n^m Q_m \supset \cdots$$

stabilizes; that is, $\alpha_n^m Q_m = \alpha_n^{m+k} Q_{m+k}$ for all $k > 0$.

(1) Assume for each n, there is $m > n$ with $\alpha_n^m = 0$. Show $\varprojlim^1 Q_n = 0$.

(2) Assume α_n^{n+1} is surjective for all n. Show $\varprojlim^1 Q_n = 0$.

(3) Assume the Q_n satisfy the Mittag-Leffler Condition. Set $P_n := \bigcap_{m \ge n} \alpha_n^m Q_m$, which is the stable submodule. Show $\alpha_n^{n+1} P_{n+1} = P_n$.

(4) Assume the Q_n satisfy the Mittag-Leffler Condition. Show $\varprojlim^1 Q_n = 0$.

Lemma (**22.7**). — *For $n \ge 0$, consider commutative diagrams with exact rows*

$$\begin{CD}
0 @>>> Q'_{n+1} @>{\gamma'_{n+1}}>> Q_{n+1} @>{\gamma_{n+1}}>> Q''_{n+1} @>>> 0\\
@. @V{\alpha'^{n+1}_n}VV @V{\alpha^{n+1}_n}VV @V{\alpha''^{n+1}_n}VV @.\\
0 @>>> Q'_n @>{\gamma'_n}>> Q_n @>{\gamma_n}>> Q''_n @>>> 0
\end{CD}$$

Then the induced sequence

$$0 \to \varprojlim Q'_n \xrightarrow{\widehat{\gamma}'} \varprojlim Q_n \xrightarrow{\widehat{\gamma}} \varprojlim Q''_n \qquad \textbf{(22.7.1)}$$

is exact; further, $\widehat{\gamma}$ is surjective if the Q'_n satisfy the Mittag-Leffler Condition.

Proof: The given commutative diagrams yield the following one:

$$\begin{CD}
0 @>>> \prod Q'_n @>{\prod \gamma'_n}>> \prod Q_n @>{\prod \gamma_n}>> \prod Q''_n @>>> 0\\
@. @V{\theta'}VV @V{\theta}VV @V{\theta''}VV @.\\
0 @>>> \prod Q'_n @>{\prod \gamma'_n}>> \prod Q_n @>{\prod \gamma_n}>> \prod Q''_n @>>> 0
\end{CD}$$

Owing to (**22.5.1**), the Snake Lemma (**5.13**) yields the exact sequence (**22.7.1**) and an injection $\operatorname{Coker} \widehat{\gamma} \hookrightarrow \varprojlim^1 Q'_n$. Assume the Q'_n satisfy the Mittag-Leffler Condition. Then $\varprojlim^1 Q'_n = 0$ by (**22.6**). So $\operatorname{Coker} \widehat{\gamma} = 0$. Thus $\widehat{\gamma}$ is surjective. □

Proposition (**22.8**). — *Let R be a ring, M a module, $F^\bullet M$ a filtration. Then*

$$\widehat{M} \xrightarrow{\sim} \varprojlim (M/F^n M).$$

Its rows are exact. So the Snake Lemma (5.13) yields this exact sequence:

$$\operatorname{Ker} G^n\alpha \to \operatorname{Ker}\alpha_{n+1} \to \operatorname{Ker}\alpha_n \to \operatorname{Coker} G^n\alpha \to \operatorname{Coker}\alpha_{n+1} \to \operatorname{Coker}\alpha_n.$$

Assume $G^\bullet\alpha$ is injective. Then $\operatorname{Ker} G^n\alpha = 0$. But $M/F^nM = 0$ for $n \ll 0$. So by induction $\operatorname{Ker}\alpha_n = 0$ for all n. Thus $\widehat{\alpha}$ is injective by (22.7) and (22.8).

Assume $G^\bullet\alpha$ is surjective, or $\operatorname{Coker} G^n\alpha = 0$. So $\operatorname{Ker}\alpha_{n+1} \to \operatorname{Ker}\alpha_n$ is surjective. But $N/F^nN = 0$ for $n \ll 0$. So by induction, $\operatorname{Coker}\alpha_n = 0$ for all n. So

$$0 \to \operatorname{Ker}\alpha_n \to M/F^nM \xrightarrow{\alpha_n} N/F^nN \to 0$$

is exact. Thus $\widehat{\alpha}$ is surjective by (22.7) and (22.8). □

Lemma (**22.27**). — *Let R be a ring, $\mathfrak{a}$ an ideal, M a module, $F^\bullet M$ an $\mathfrak{a}$-filtration. Assume R is complete, M is separated, and $F^nM = M$ for $n \ll 0$. Assume $G^\bullet M$ is module finite over $G^\bullet R$. Then M is complete, and is module finite over R.*

Proof: Take finitely many generators μ_i of $G^\bullet M$, and replace them by their homogeneous components. Set $n_i := \deg(\mu_i)$. Lift μ_i to $m_i \in F^{n_i}M$.

Filter R $\mathfrak{a}$-adically. Set $E := \bigoplus_i R[-n_i]$. Filter E with $F^nE := \bigoplus_i F^n(R[-n_i])$. Then $F^nE = E$ for $n \ll 0$. Define $\alpha\colon E \to M$ by sending $1 \in R[-n_i]$ to $m_i \in M$. Then $\alpha F^nE \subset F^nM$ for all n. Also, $G^\bullet\alpha\colon G^\bullet E \to G^\bullet M$ is surjective as the μ_i generate. So $\widehat{\alpha}$ is surjective by (22.26).

Form the following canonical commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\kappa_E} & \widehat{E} \\ \alpha\downarrow & & \downarrow\widehat{\alpha} \\ M & \xrightarrow{\kappa_M} & \widehat{M} \end{array}$$

As R is complete, $\kappa_R\colon R \to \widehat{R}$ is surjective by (22.1); hence, κ_E is surjective. Thus κ_M is surjective; that is, M is complete. As M is separated, κ_M is injective by (22.4). So κ_M is bijective. So α is surjective. Thus M is module finite. □

Exercise (22.28) (*Nakayama's Lemma for a complete ring*). — Let R be a ring, $\mathfrak{a}$ an ideal, and M a module. Assume R is complete, and M separated. Show $m_1, \dots, m_n \in M$ generate assuming their images $m_1', \dots, m_n'$ in $M/\mathfrak{a}M$ generate.

Proposition (**22.29**). — *Let R be a ring, $\mathfrak{a}$ an ideal, and M a module. Assume R is complete, and M separated. Assume $G^\bullet M$ is a Noetherian $G^\bullet R$-module. Then M is a Noetherian R-module, and every submodule N is complete.*

Proof: Let $F^\bullet M$ denote the $\mathfrak{a}$-adic filtration, and $F^\bullet N$ the induced filtration: $F^nN := N \cap F^nM$. Then N is separated, and $F^nN = N$ for $n \ll 0$. Further, $G^\bullet N \subset G^\bullet M$. However, $G^\bullet M$ is Noetherian. So $G^\bullet N$ is module finite. Thus N is complete and is module finite over R by (22.27). Thus M is Noetherian. □

Theorem (**22.30**). — *Let R be a ring, $\mathfrak{a}$ an ideal. If R is Noetherian, so is $\widehat{R}$.*

Proof: Assume R is Noetherian. Then $G^\bullet R$ is algebra finite over $R/\mathfrak{a}$ by (20.12), so Noetherian by the Hilbert Basis Theorem, (16.12). But $G^\bullet R = G^\bullet\widehat{R}$ by (22.11). Thus $\widehat{R}$ is Noetherian by (22.29) with $\widehat{R}$ for R and $\widehat{R}$ for M. □

23. Appendix: Cohen–Macaulayness

Exercise (23.25). — Let $R \to R'$ be a flat map of Noetherian rings, $\mathfrak{a} \subset R$ an ideal, M a finitely generated R-module, and $x_1, \ldots, x_r$ an M-sequence in $\mathfrak{a}$. Set $M' := M \otimes_R R'$. Assume $M'/\mathfrak{a}M' \neq 0$. Show $x_1, \ldots, x_r$ is an M'-sequence in $\mathfrak{a}R'$.

Exercise (23.26). — Let R be a Noetherian ring, $\mathfrak{a}$ an ideal, and M a finitely generated module with $M/\mathfrak{a}M \neq 0$. Let $x_1, \ldots, x_r$ be an M-sequence in $\mathfrak{a}$ and $\mathfrak{p} \in \operatorname{Supp}(M/\mathfrak{a}M)$. Prove the following statements:

(1) $x_1/1, \ldots, x_r/1$ is an $M_\mathfrak{p}$-sequence in $\mathfrak{a}_\mathfrak{p}$, and

(2) $\operatorname{depth}(\mathfrak{a}, M) \leq \operatorname{depth}(\mathfrak{a}_\mathfrak{p}, M_\mathfrak{p})$.

(23.27) (*Finished Sequences*). — Let R be a ring, $\mathfrak{a}$ an ideal, M a nonzero module. We say an M-sequence in $\mathfrak{a}$ is **finished in** $\mathfrak{a}$, if it can not be lengthened in $\mathfrak{a}$.

In particular, a sequence of length 0 is finished in $\mathfrak{a}$ if there are no nonzerodivisors on M in $\mathfrak{a}$; that is, $\mathfrak{a} \subset \operatorname{z.div}(M)$.

An M-sequence in $\mathfrak{a}$ can, plainly, be lengthened until finished in $\mathfrak{a}$ provided $\operatorname{depth}(\mathfrak{a}, M)$ is finite. It is finite if R is Noetherian, M is finitely generated, and $M/\mathfrak{a}M \neq 0$, as then $\operatorname{depth}(\mathfrak{a}, M) \leq \operatorname{depth}(M_\mathfrak{p})$ for any $\mathfrak{p} \in \operatorname{Supp}(M/\mathfrak{a}M)$ by (23.26)(2) and $\operatorname{depth}(M_\mathfrak{p}) \leq \dim(M_\mathfrak{p})$ by (23.5)(3) and $\dim(M_\mathfrak{p}) < \infty$ by (21.4).

Proposition (23.28). — *Let R be a Noetherian ring, $\mathfrak{a}$ an ideal, and M a finitely generated module. Assume $M/\mathfrak{a}M \neq 0$. Let $x_1, \ldots, x_m$ be a finished M-sequence in $\mathfrak{a}$. Then $m = \operatorname{depth}(\mathfrak{a}, M)$.*

Proof. — Let $y_1, \ldots, y_n$ be a second finished M-sequence in $\mathfrak{a}$. Say $m \leq n$. Induct on m. Suppose $m = 0$. Then $\mathfrak{a} \subset \operatorname{z.div}(M)$. Hence $n = 0$ too. Now, suppose $m \geq 1$.

Set $M_i := M/\langle x_1, \ldots, x_i \rangle M$ and $N_j := M/\langle y_1, \ldots, y_j \rangle M$ for all i, j. Set

$$U := \bigcup_{i=0}^{m-1} \operatorname{z.div}(M_i) \cup \bigcup_{j=0}^{n-1} \operatorname{z.div}(N_j).$$

Then U is equal to the union of all associated primes of M_i for $i < m$ and of N_j for $j < n$ by (17.15). And these primes are finite in number by (17.21). Suppose $\mathfrak{a} \subset U$. Then $\mathfrak{a}$ lies in one of the primes, say $\mathfrak{p} \in \operatorname{Ass}(M_i)$, by (3.19). But $x_{i+1} \in \mathfrak{a} - \operatorname{z.div}(M_i)$ and $\mathfrak{a} \subset \mathfrak{p} \subset \operatorname{z.div}(M_i)$, a contradiction. Thus $\mathfrak{a} \not\subset U$.

Take $z \in \mathfrak{a} - U$. Then $z \notin \operatorname{z.div}(M_i)$ for $i < m$ and $z \notin \operatorname{z.div}(N_j)$ for $j < n$. Now, $\mathfrak{a} \subset \operatorname{z.div}(M_m)$ by finishedness. So $\mathfrak{a} \subset \mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Ass}(M_m)$ by (17.26). But $M_m = M_{m-1}/x_m M_{m-1}$. Moreover, x_m and z are nonzerodivisors on M_{m-1}. Also $x_m, z \in \mathfrak{a} \subset \mathfrak{q}$. So $\mathfrak{q} \in \operatorname{Ass}(M_{m-1}/zM_{m-1})$ by (17.27). Hence

$$\mathfrak{a} \subset \operatorname{z.div}(M/\langle x_1, \ldots, x_{m-1}, z \rangle M).$$

Hence $x_1, \ldots, x_{m-1}, z$ is finished in $\mathfrak{a}$. Similarly, $y_1, \ldots, y_{n-1}, z$ is finished in $\mathfrak{a}$. Thus we may replace both x_m and y_n by z.

By (23.6)(2), we may move z to the front of both sequences. Thus we may assume $x_1 = y_1 = z$. Then $M_1 = N_1$. Further, $x_2, \ldots, x_m$ and $y_2, \ldots, y_n$ are finished M_1-sequences in $\mathfrak{a}$. So by induction, $m - 1 = n - 1$. Thus $m = n$. □

Exercise (23.29). — Let R be a Noetherian ring, $\mathfrak{a}$ an ideal, and M a finitely generated module with $M/\mathfrak{a}M \neq 0$. Let $x \in \mathfrak{a}$ be a nonzerodivisor on M. Show

$$\operatorname{depth}(\mathfrak{a}, M/xM) = \operatorname{depth}(\mathfrak{a}, M) - 1.$$

Definition **(23.40)**. — Let R be a Noetherian ring, and M a finitely generated module. We call M **Cohen–Macaulay** if $M_{\mathfrak{m}}$ is a Cohen–Macaulay $R_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m} \in \operatorname{Supp}(M)$. It is equivalent that $M_{\mathfrak{p}}$ be a Cohen–Macaulay $R_{\mathfrak{p}}$-module for every $\mathfrak{p} \in \operatorname{Supp}(M)$, because if $\mathfrak{p}$ lies in the maximal ideal $\mathfrak{m}$, then $R_{\mathfrak{p}}$ is the localization of $R_{\mathfrak{m}}$ at the prime ideal $\mathfrak{p}R_{\mathfrak{m}}$ by **(11.28)**, and hence $R_{\mathfrak{p}}$ is Cohen–Macaulay if $R_{\mathfrak{m}}$ is by **(23.39)**.

We say R is **Cohen–Macaulay** if R is a Cohen–Macaulay R-module.

Proposition **(23.41)**. — *Let R be a Noetherian ring. Then R is Cohen–Macaulay if and only if the polynomial ring $R[X]$ is Cohen–Macaulay.*

Proof: First, assume $R[X]$ is Cohen–Macaulay. Given a prime $\mathfrak{p}$ of R, set $\mathfrak{P} := \mathfrak{p}R[X] + \langle X \rangle$. Then $\mathfrak{P}$ is prime in $R[X]$ by **(2.18)**. Now, $R[X]/\langle X \rangle = R$ and $\mathfrak{P}/\langle X \rangle = \mathfrak{p}$ owing to **(1.8)**; hence, $R_{\mathfrak{P}} = R_{\mathfrak{p}}$ by **(11.29)**(1). Further, **(12.22)** yields $(R[X]/\langle X \rangle)_{\mathfrak{P}} = R[X]_{\mathfrak{P}}/\langle X \rangle R[X]_{\mathfrak{P}}$. Hence $R[X]_{\mathfrak{P}}/\langle X \rangle R[X]_{\mathfrak{P}} = R_{\mathfrak{p}}$. But $R[X]_{\mathfrak{P}}$ is Cohen–Macaulay by **(23.40)**, and X is plainly a nonzerodivisor; so $R_{\mathfrak{p}}$ is Cohen–Macaulay by **(23.30)**. Thus R is Cohen–Macaulay.

Conversely, assume R is Cohen–Macaulay. Given a maximal ideal $\mathfrak{M}$ of $R[X]$, set $\mathfrak{m} := \mathfrak{M} \cap R$. Then $R[X]_{\mathfrak{M}} = (R[X]_{\mathfrak{m}})_{\mathfrak{M}}$ by **(11.29)**(1), and $R[X]_{\mathfrak{m}} = R_{\mathfrak{m}}[X]$ by **(11.30)**. But $R_{\mathfrak{m}}$ is Cohen–Macaulay. Thus, to show $R[X]_{\mathfrak{M}}$ is Cohen–Macaulay, replace R by $R_{\mathfrak{m}}$, and so assume R is local with maximal ideal $\mathfrak{m}$.

As $\mathfrak{M}(R/\mathfrak{m})[X]$ is maximal, it contains a nonzero polynomial $\overline{f}$. As $R/\mathfrak{m}$ is a field, we may take $\overline{f}$ monic. Lift $\overline{f}$ to a monic polynomial $f \in \mathfrak{M}$. Set $B := R[X]/\langle f \rangle$. Then B is a free, module-finite extension of R by **(10.2[illegible])**. So $\dim(R) = \dim(B)$ by **(15.12)**. Plainly $\dim(B) \ge \dim(B_{\mathfrak{M}})$. So $\dim(R) \ge \dim(B_{\mathfrak{M}})$.

Further, B is flat over R by **(9.7)**. And $B_{\mathfrak{M}}$ is flat over B by **(12.21)**. So $B_{\mathfrak{M}}$ is flat over R by **(9.12)**. So any R-sequence in $\mathfrak{m}$ is a $B_{\mathfrak{M}}$-sequence by **(23.25)** as $B_{\mathfrak{M}}/\mathfrak{m}B_{\mathfrak{M}} \neq 0$. Hence $\operatorname{depth}(B_{\mathfrak{M}}) \ge \operatorname{depth}(R)$.

But $\operatorname{depth}(R) = \dim(R)$ and $\dim(R) \ge \dim(B_{\mathfrak{M}})$. So $\operatorname{depth}(B_{\mathfrak{M}}) \ge \dim(B_{\mathfrak{M}})$. But the opposite inequality holds by **(23.5)**. Thus $B_{\mathfrak{M}}$ is Cohen–Macaulay. But $B_{\mathfrak{M}} = R[X]_{\mathfrak{M}}/\langle f \rangle R[X]_{\mathfrak{M}}$ by **(12.22)**. And f is monic, so a nonzerodivisor. So $R[X]_{\mathfrak{M}}$ is Cohen–Macaulay by **(23.30)**. Thus $R[X]$ is Cohen–Macaulay. □

Definition **(23.42)**. — A ring R is called **universally catenary** if every finitely generated R-algebra is catenary.

Theorem **(23.43)**. — *A Cohen–Macaulay ring R is universally catenary.*

Proof: Clearly any quotient of a catenary ring is catenary, as chains of primes can be lifted by **(1.9)**. So it suffices to prove that, for any n, the polynomial ring P in n variables over R is catenary.

Notice P is Cohen–Macaulay by induction on n, as $P = R$ if $n = 0$, and the induction step holds by **(23.41)**. Now, given nested primes $\mathfrak{q} \subset \mathfrak{p}$ in P, put $\mathfrak{p}$ in a maximal ideal $\mathfrak{m}$. Then any chain of primes from $\mathfrak{q}$ to $\mathfrak{p}$ corresponds to a chain from $\mathfrak{q}P_{\mathfrak{m}}$ to $\mathfrak{p}P_{\mathfrak{m}}$ by **(11.20)**. But $P_{\mathfrak{m}}$ is Cohen–Macaulay, so catenary by **(23.37)**. Thus the assertion holds. □

Example **(23.44)**. — Trivially, a field is Cohen–Macaulay. Plainly, a domain of dimension 1 is Cohen–Macaulay. By **(23.20)**, a normal domain of dimension 2 is Cohen–Macaulay. Thus these rings are all universally catenary by **(23.43)**. In particular, we recover **(15.16)**.

(1) *M is an invertible fractional ideal.*
(2) *M is an invertible abstract module.*
(3) *M is a projective abstract module.*

Proof: Assume (1). Then M is locally principal by **(25.13)**. So **(25.6)** yields $M \otimes M^{-1} = MM^{-1}$ by (1). But $MM^{-1} = 1$. Thus (2) holds.

If (2) holds, then M is locally free of rank 1 by **(25.18)**; so **(13.51)** yields (3).

Finally, assume (3). By **(5.23)**, there's an M' with $M \oplus M' \simeq R^{\oplus\Lambda}$. Let $\rho\colon R^{\oplus\Lambda} \to M$ be the projection, and set $x_\lambda := \rho(e_\lambda)$ where e_λ is the standard basis vector. Define $\varphi_\lambda\colon M \hookrightarrow R^{\oplus\Lambda} \to R$ to be the composition of the injection with the projection φ_λ on the λth factor. Then given $x \in M$, we have $\varphi_\lambda(x) = 0$ for almost all λ and $x = \sum_{\lambda\in\Lambda} \varphi_\lambda(x) x_\lambda$.

Fix a nonzero $y \in M$. For $\lambda \in \Lambda$, set $q_\lambda := \frac{1}{y}\varphi_\lambda(y) \in \mathrm{Frac}(R)$. Set $N := \sum Rq_\lambda$. Given any nonzero $x \in M$, say $x = a/b$ and $y = c/d$ with $a, b, c, d \in R$. Then $a, c \in M$; whence, $ad\varphi_\lambda(y)\varphi_\lambda(ac) = bc\varphi_\lambda(x)$. Thus $xq_\lambda = \varphi_\lambda(x) \in R$. Hence $M \cdot N \subset R$. But $y = \sum \varphi_\lambda(y) y_\lambda$; so $1 = y_\lambda q_\lambda$. Thus $M \cdot N = R$. Thus (1) holds. □

Theorem **(25.20)**. — *Let R be a domain. Then the following are equivalent:*

(1) *R is a Dedekind domain or a field.*
(2) *Every nonzero ordinary ideal $\mathfrak{a}$ is invertible.*
(3) *Every nonzero ordinary ideal $\mathfrak{a}$ is projective.*
(4) *Every nonzero ordinary ideal $\mathfrak{a}$ is finitely generated and flat.*

Proof: Assume R is not a field; otherwise, (1)–(4) hold trivially.

If R is Dedekind, then **(25.14)** yields (2) since $\mathfrak{a} = (\mathfrak{a} : R)$.

Assume (2). Then $\mathfrak{a}$ is finitely generated by **(25.10)**. Thus R is Noetherian. Let $\mathfrak{p}$ be any nonzero prime of R. Then by hypothesis, $\mathfrak{p}$ is invertible. So by **(25.13)**, $\mathfrak{p}$ is locally principal. So $R_\mathfrak{p}$ is a DVR by **(23.10)**. Hence R is Dedekind by **(24.7)**. Thus (1) holds. Thus (1) and (2) are equivalent.

By **(25.19)**, (2) and (3) are equivalent. But (2) implies that R is Noetherian by **(25.10)**. Thus (3) and (4) are equivalent by **(16.19)** and **(13.51)**. □

Theorem **(25.21)**. — *Let R be a Noetherian domain, but not a field. Then R is Dedekind if and only if every torsionfree module is flat.*

Proof: (Of course, as R is a domain, every flat module is torsionfree by **(9.28)**.)

Assume R is Dedekind. Let M be a torsionfree module, $\mathfrak{m}$ a maximal ideal. Let's see that $M_\mathfrak{m}$ is torsionfree over $R_\mathfrak{m}$. Let $z \in R_\mathfrak{m}$ be nonzero, and say $z = x/s$ with $x, s \in R$ and $s \notin \mathfrak{m}$. Then $\mu_x\colon M \to M$ is injective as M is torsionfree. So $\mu_x\colon M_\mathfrak{m} \to M_\mathfrak{m}$ is injective by the Exactness of Localization. But $\mu_{x/s} = \mu_x \mu_{1/s}$ and $\mu_{1/s}$ is invertible. So $\mu_{x/s}$ is injective. Thus $M_\mathfrak{m}$ is torsionfree.

Since R is Dedekind, $R_\mathfrak{m}$ is a DVR by **(24.7)**, so a PID by **(24.1)**. Hence $M_\mathfrak{m}$ is flat over $R_\mathfrak{m}$ by **(9.28)**. But $\mathfrak{m}$ is arbitrary. Thus by **(13.46)**, M is flat over R.

Conversely, assume every torsionfree module is flat. In particular, every nonzero ordinary ideal is flat. But R is Noetherian. Thus R is Dedekind by **(25.20)**. □

(25.22) (*The Picard Group*). — Let R be a ring. We denote the collection of isomorphism classes of invertible modules by $\mathrm{Pic}(R)$. By **(25.17)**, every invertible module is finitely generated, so isomorphic to a quotient of R^n for some integer n. Hence, $\mathrm{Pic}(R)$ is a set. Further, $\mathrm{Pic}(R)$ is, clearly, a group under tensor product

26. Arbitrary Valuation Rings

A valuation ring is a subring of a field such that the reciprocal of any element outside the subring lies in it. Valuation rings are normal local domains. They are maximal under *domination of local rings*; that is, one contains the other, and the inclusion map is a local homomorphism. Given any domain, its normalization is equal to the intersection of all the valuation rings containing it. Given a 1-dimensional Noetherian domain and a finite extension of its fraction field with a proper subring containing the domain, that subring too is 1-dimensional and Noetherian, this is the Krull–Akizuki Theorem. So normalizing a Dedekind domain in any finite extension of its fraction field yields another Dededind domain.

Definition **(26.1)**. — A subring V of a field K is said to be a **valuation ring** of K if, whenever $z \in K - V$, then $1/z \in V$.

Proposition **(26.2)**. — *Let V be a valuation ring of a field K, and set*

$$\mathfrak{m} := \{1/z \mid z \in K - V\} \cup \{0\}.$$

Then V is local, $\mathfrak{m}$ is its maximal ideal, and K is its fraction field.

Proof: Clearly $\mathfrak{m} = V - V^\times$. Let's show $\mathfrak{m}$ is an ideal. Take a nonzero $a \in V$ and nonzero $x, y \in \mathfrak{m}$. Suppose $ax \notin \mathfrak{m}$. Then $1/ax \in V$. So $a(1/ax) \in V$. So $1/x \in V$. So $x \in V^\times$, a contradiction. Thus $ax \in \mathfrak{m}$. Now, by hypothesis, either $x/y \in V$ or $y/x \in V$. Say $y/x \in V$. Then $1 + (y/x) \in V$. So $x + y = (1 + (y/x))x \in \mathfrak{m}$. Thus $\mathfrak{m}$ is an ideal. Hence V is local and $\mathfrak{m}$ is its maximal ideal by (3.6). Finally, K is its fraction field, because whenever $z \in K - V$, then $1/z \in V$. □

Exercise (26.3). — Let V be a domain. Show that V is a valuation ring if and only if, given any two ideals $\mathfrak{a}$ and $\mathfrak{b}$, either $\mathfrak{a}$ lies in $\mathfrak{b}$ or $\mathfrak{b}$ lies in $\mathfrak{a}$.

Exercise (26.4). — Let V be a valuation ring of K, and $V \subset W \subset K$ a subring. Prove that W is also a valuation ring of K, that its maximal ideal $\mathfrak{p}$ lies in V, that $V/\mathfrak{p}$ is a valuation ring of the field $W/\mathfrak{p}$, and that $W = V_{\mathfrak{p}}$.

Exercise (26.5). — Prove that a valuation ring V is normal.

Lemma **(26.6)**. — *Let R be a domain, $\mathfrak{a}$ an ideal, $K := \mathrm{Frac}(R)$, and $x \in K^\times$. Then either $1 \notin \mathfrak{a}R[x]$ or $1 \notin \mathfrak{a}R[1/x]$.*

Proof: Assume $1 \in \mathfrak{a}R[x]$ and $1 \in \mathfrak{a}R[1/x]$. Then there are equations

$$1 = a_0 + \cdots + a_n x^n \quad \text{and} \quad 1 = b_0 + \cdots + b_m/x^m \quad \text{with all} \quad a_i, b_j \in \mathfrak{a}.$$

Assume n, m minimal and $m \le n$. Multiply through by $1 - b_0$ and $a_n x^n$, getting

$$1 - b_0 = (1 - b_0)a_0 + \cdots + (1 - b_0)a_n x^n \quad \text{and}$$
$$(1 - b_0)a_n x^n = a_n b_1 x^{n-1} + \cdots + a_n b_m x^{n-m}.$$

Combine the latter equations, getting

$$1 - b_0 = (1 - b_0)a_0 + \cdots + (1 - b_0)a_{n-1}x^{n-1} + a_n b_1 x^{n-1} + \cdots + a_n b_m x^{n-m}.$$

Simplify, getting an equation of the form $1 = c_0 + \cdots + c_{n-1}x^{n-1}$ with $c_i \in \mathfrak{a}$, which contradicts the minimality of n. □

(26.7) (*Domination*). — Let A, B be local rings, and $\mathfrak{m}$, $\mathfrak{n}$ their maximal ideals. We say B **dominates** A if $B \supset A$ and $\mathfrak{n} \cap A = \mathfrak{m}$; in other words, the inclusion map $\varphi\colon A \hookrightarrow B$ is a local homomorphism.

Proposition **(26.8)**. — *Let K be a field, A any local subring. Then A is dominated by a valuation ring V of K with algebraic residue field extension.*

Proof: Let $\mathfrak{m}$ be the maximal ideal of A. Let $\mathcal{S}$ be the set of pairs $(R, \mathfrak{n})$ where $R \subset K$ is a subring containing A and where $\mathfrak{n} \subset R$ is a maximal ideal with $\mathfrak{n} \cap A = \mathfrak{m}$ and with $R/\mathfrak{n}$ an algebraic extension of $A/\mathfrak{m}$. Then $(A, \mathfrak{m}) \in \mathcal{S}$. Order $\mathcal{S}$ as follows: $(R, \mathfrak{n}) \le (R', \mathfrak{n}')$ if $R \subset R'$ and $\mathfrak{n} = \mathfrak{n}' \cap R$. Let $(R_\lambda, \mathfrak{n}_\lambda)$ form a totally ordered subset. Set $B := \bigcup R_\lambda$ and $\mathfrak{N} = \bigcap \mathfrak{n}_\lambda$. Plainly $\mathfrak{N} \cap R_\lambda = \mathfrak{n}_\lambda$ and $B/\mathfrak{N} = \bigcap R_\lambda/\mathfrak{n}_\lambda$ for all λ. So any $y \in B/\mathfrak{N}$ is in $R_\lambda/\mathfrak{n}_\lambda$ for some λ. Hence $B/\mathfrak{N}$ is a field and is algebraic over $A/\mathfrak{m}$. Thus by Zorn's Lemma, $\mathcal{S}$ has a maximal element, say $(V, \mathfrak{M})$.

For any nonzero $x \in K$, set $V' := V[x]$ and $V'' := V[1/x]$. By **(26.6)**, either $1 \notin \mathfrak{M}V'$ or $1 \notin \mathfrak{M}V''$. Say $1 \notin \mathfrak{M}V'$. Then $\mathfrak{M}V'$ is proper, so it is contained in a maximal ideal $\mathfrak{M}'$ of V'. Since $\mathfrak{M}' \cap V \supset \mathfrak{M}$ and $V \cap \mathfrak{M}'$ is proper, $\mathfrak{M}' \cap V = \mathfrak{M}$. Further $V'/\mathfrak{M}'$ is generated as a ring over $V/\mathfrak{M}$ by the residue x' of x. Hence x' is algebraic over $V/\mathfrak{M}$; otherwise, $V'/\mathfrak{M}'$ would be a polynomial ring, so not a field. Hence $(V', \mathfrak{M}') \in \mathcal{S}$, and $(V', \mathfrak{M}') \ge (V, \mathfrak{M})$. By maximality, $V = V'$; so $x \in V$. Thus V is a valuation ring of K. So V is local, and $\mathfrak{M}$ is its unique maximal ideal. Finally, $(V, \mathfrak{M}) \in \mathcal{S}$; so V dominates A with algebraic residue field extension. □

Exercise **(26.9)**. — Let K be a field, $\mathcal{S}$ the set of local subrings ordered by domination. Show that the valuation rings of K are the maximal elements of $\mathcal{S}$.

Theorem **(26.10)**. — *Let R be any subring of a field K. Then the integral closure $\overline{R}$ of R in K is the intersection of all valuation rings V of K containing R. Further, if R is local, then the V dominating R with algebraic residue field extension suffice.*

Proof: Every valuation ring V is normal by **(26.5)**. So if $V \supset R$, then $V \supset \overline{R}$. Thus $\left(\bigcap_{V \supset R} V\right) \supset \overline{R}$.

To prove the opposite inclusion, take any $x \in K - \overline{R}$. To find a valuation ring V with $V \supset R$ and $x \notin V$, set $y := 1/x$. If $1/y \in R[y]$, then for some n,

$$1/y = a_0 y^n + a_1 y^{n-1} + \cdots + a_n \quad \text{with} \quad a_\lambda \in R.$$

Multiplying by x^n yields $x^{n+1} - a_n x^n - \cdots - a_0 = 0$. So $x \in \overline{R}$, a contradiction.

Thus $1 \notin yR[y]$. So there is a maximal ideal $\mathfrak{m}$ of $R[y]$ containing y. Then the composition $R \to R[y] \to R[y]/\mathfrak{m}$ is surjective as $y \in \mathfrak{m}$. Its kernel is $\mathfrak{m} \cap R$, so $\mathfrak{m} \cap R$ is a maximal ideal of R. By **(26.8)**, there is a valuation ring V that dominates $R[y]_{\mathfrak{m}}$ with algebraic residue field extension; whence, if R is local, then V also dominates R, and the residue field of $R[y]_{\mathfrak{m}}$ is equal to that of R. But $y \in \mathfrak{m}$; so $x = 1/y \notin V$, as desired. □

(26.11) (*Valuations*). — We call an additive abelian group Γ **totally ordered** if Γ has a subset Γ_+ that is closed under addition and satisfies $\Gamma_+ \sqcup \{0\} \sqcup -\Gamma_+ = \Gamma$.

Given $x, y \in \Gamma$, write $x > y$ if $x - y \in \Gamma_+$. Note that either $x > y$ or $x = y$ or $y > x$. Note that, if $x > y$, then $x + z > y + z$ for any $z \in \Gamma$.

Let V be a domain, and set $K := \mathrm{Frac}(V)$ and $\Gamma := K^\times/V^\times$. Write the group Γ additively, and let $v\colon K^\times \to \Gamma$ be the quotient map. It is a homomorphism:

$$v(xy) = v(x) + v(y). \qquad \textbf{(26.11.1)}$$

(2) Let $\mathfrak{a}$ be comaximal to both $\mathfrak{b}$ and $\mathfrak{b}'$. Prove $\mathfrak{a}$ is also comaximal to $\mathfrak{b}\mathfrak{b}'$.
(3) Let $\mathfrak{a}$, $\mathfrak{b}$ be comaximal, and $m, n \geq 1$. Prove $\mathfrak{a}^m$ and $\mathfrak{b}^n$ are comaximal.
(4) Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be pairwise comaximal. Prove

(a) $\mathfrak{a}_1$ and $\mathfrak{a}_2 \cdots \mathfrak{a}_n$ are comaximal;
(b) $\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n = \mathfrak{a}_1 \cdots \mathfrak{a}_n$;
(c) $R/(\mathfrak{a}_1 \cdots \mathfrak{a}_n) \xrightarrow{\sim} \prod (R/\mathfrak{a}_i)$.

Solution: To prove (1)(a), note that always $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$. Conversely, $\mathfrak{a} + \mathfrak{b} = R$ implies $x + y = 1$ with $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$. So given $z \in \mathfrak{a} \cap \mathfrak{b}$, we have $z = xz + yz \in \mathfrak{a}\mathfrak{b}$.

To prove (1)(b), form the map $R \to R/\mathfrak{a} \times R/\mathfrak{b}$ that carries an element to its pair of residues. The kernel is $\mathfrak{a} \cap \mathfrak{b}$, which is $\mathfrak{a}\mathfrak{b}$ by (1). So we have an injection

$$\varphi\colon R/\mathfrak{a}\mathfrak{b} \hookrightarrow R/\mathfrak{a} \times R/\mathfrak{b}.$$

To show that φ is surjective, take any element $(\bar{x}, \bar{y})$ in $R/\mathfrak{a} \times R/\mathfrak{b}$. Say $\bar{x}$ and $\bar{y}$ are the residues of x and y. Since $\mathfrak{a} + \mathfrak{b} = R$, we can find $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ such that $a + b = y - x$. Then $\varphi(x + a) = (\bar{x}, \bar{y})$, as desired. Thus (1) holds.

To prove (2), note that

$$R = (\mathfrak{a} + \mathfrak{b})(\mathfrak{a} + \mathfrak{b}') = (\mathfrak{a}^2 + \mathfrak{b}\mathfrak{a} + \mathfrak{a}\mathfrak{b}') + \mathfrak{b}\mathfrak{b}' \subseteq \mathfrak{a} + \mathfrak{b}\mathfrak{b}' \subseteq R.$$

To prove (3), note that (2) implies $\mathfrak{a}$ and $\mathfrak{b}^n$ are comaximal for any $n \geq 1$ by induction on n. Hence, $\mathfrak{b}^n$ and $\mathfrak{a}^m$ are comaximal for any $m \geq 1$.

To prove (4)(a), assume $\mathfrak{a}_1$ and $\mathfrak{a}_2 \cdots \mathfrak{a}_{n-1}$ are comaximal by induction on n. By hypothesis, $\mathfrak{a}_1$ and $\mathfrak{a}_n$ are comaximal. Thus (2) yields (a).

To prove (4)(b) and (4)(c), again proceed by induction on n. Thus (1) yields

$$\mathfrak{a}_1 \cap (\mathfrak{a}_2 \cap \cdots \cap \mathfrak{a}_n) = \mathfrak{a}_1 \cap (\mathfrak{a}_2 \cdots \mathfrak{a}_n) = \mathfrak{a}_1\mathfrak{a}_2 \cdots \mathfrak{a}_n;$$
$$R/(\mathfrak{a}_1 \cdots \mathfrak{a}_n) \xrightarrow{\sim} R/\mathfrak{a}_1 \times R/(\mathfrak{a}_2 \cdots \mathfrak{a}_n) \xrightarrow{\sim} \prod (R/\mathfrak{a}_i). \qquad \square$$

Exercise **(1.15)**. — First, given a prime number p and a $k \geq 1$, find the idempotents in $\mathbb{Z}/\langle p^k \rangle$. Second, find the idempotents in $\mathbb{Z}/\langle 12 \rangle$. Third, find the number of idempotents in $\mathbb{Z}/\langle n \rangle$ where $n = \prod_{i=1}^{N} p_i^{n_i}$ with p_i distinct prime numbers.

Solution: First, let $m \in \mathbb{Z}$ be idempotent modulo p^k. Then $m(m-1)$ is divisible by p^k. So either m or $m - 1$ is divisible by p^k, as m and $m - 1$ have no common prime divisor. Hence 0 and 1 are the only idempotents in $\mathbb{Z}/\langle p^k \rangle$.

Second, since $-3 + 4 = 1$, the Chinese Remainder Theorem (**1.14**) yields

$$\mathbb{Z}/\langle 12 \rangle = \mathbb{Z}/\langle 3 \rangle \times \mathbb{Z}/\langle 4 \rangle.$$

Hence m is idempotent modulo 12 if and only if m is idempotent modulo 3 and modulo 4. By the previous case, we have the following possibilities:

$$\begin{aligned}
m &\equiv 0 \pmod{3} \quad \text{and} \quad m \equiv 0 \pmod{4};\\
m &\equiv 1 \pmod{3} \quad \text{and} \quad m \equiv 1 \pmod{4};\\
m &\equiv 1 \pmod{3} \quad \text{and} \quad m \equiv 0 \pmod{4};\\
m &\equiv 0 \pmod{3} \quad \text{and} \quad m \equiv 1 \pmod{4}.
\end{aligned}$$

Therefore, $m \equiv 0, 1, 4, 9 \pmod{12}$.

Third, for each i, the two numbers $p_1^{n_1} \cdots p_{i-1}^{n_{i-1}}$ and $p_i^{n_i}$ have no common prime divisor. Hence some linear combination is equal to 1 by the Euclidean Algorithm. So the principal ideals they generate are comaximal. Hence by induction on N, the

Chinese Remainder Theorem yields

$$\mathbb{Z}/\langle n\rangle = \prod_{i=1}^{N} \mathbb{Z}/\langle p_i^{n_i}\rangle.$$

So m is idempotent modulo n if and only if m is idempotent modulo p^{n_i} for all i; hence, if and only if m is 0 or 1 modulo p^{n_i} for all i by the first case. Thus there are 2^N idempotents in $\mathbb{Z}/\langle n\rangle$. □

Exercise **(1.16)**. — Let $R := R' \times R''$ be a **product** of rings, $\mathfrak{a} \subset R$ an ideal. Show $\mathfrak{a} = \mathfrak{a}' \times \mathfrak{a}''$ with $\mathfrak{a}' \subset R'$ and $\mathfrak{a}'' \subset R''$ ideals. Show $R/\mathfrak{a} = (R'/\mathfrak{a}') \times (R''/\mathfrak{a}'')$.

Solution: Set $\mathfrak{a}' := \{x' \mid (x', 0) \in \mathfrak{a}\}$ and $\mathfrak{a}'' := \{x'' \mid (0, x'') \in \mathfrak{a}\}$. Clearly $\mathfrak{a}' \subset R'$ and $\mathfrak{a}'' \subset R''$ are ideals. Clearly,

$$\mathfrak{a} \supset \mathfrak{a}' \times 0 + 0 \times \mathfrak{a}'' = \mathfrak{a}' \times \mathfrak{a}''.$$

The opposite inclusion holds, because if $\mathfrak{a} \ni (x', x'')$, then

$$\mathfrak{a} \ni (x', x'') \cdot (1, 0) = (x', 0) \quad \text{and} \quad \mathfrak{a} \ni (x', x'') \cdot (0, 1) = (0, x'').$$

Finally, the equation $R/\mathfrak{a} = (R/\mathfrak{a}') \times (R/\mathfrak{a}'')$ is now clear from the construction of the residue class ring. □

Exercise **(1.17)**. — Let R be a ring, and e, e' idempotents. (See (10.7) also.)

(1) Set $\mathfrak{a} := \langle e\rangle$. Show $\mathfrak{a}$ is **idempotent**; that is, $\mathfrak{a}^2 = \mathfrak{a}$.
(2) Let $\mathfrak{a}$ be a principal idempotent ideal. Show $\mathfrak{a}\langle f\rangle$ with f idempotent.
(3) Set $e'' := e + e' - ee'$. Show $\langle e, e'\rangle = \langle e''\rangle$ and e'' is idempotent.
(4) Let $e_1, \ldots, e_r$ be idempotents. Show $\langle e_1, \ldots, e_r\rangle = \langle f\rangle$ with f idempotent.
(5) Assume R is Boolean. Show every finitely generated ideal is principal.

Solution: For (1), note $\mathfrak{a}^2 = \langle e^2\rangle$ since $\mathfrak{a} = \langle e\rangle$. But $e^2 = e$. Thus (1) holds.

For (2), say $\mathfrak{a} = \langle g\rangle$. Then $\mathfrak{a}^2 = \langle g^2\rangle$. But $\mathfrak{a}^2 = \mathfrak{a}$. So $g = xg^2$ for some x. Set $f := xg$. Then $f \in \mathfrak{a}$; so $\langle f\rangle \subset \mathfrak{a}$. And $g = fg$. So $\mathfrak{a} \subset \langle f\rangle$. Thus (2) holds.

For (3), note $\langle e''\rangle \subset \langle e, e'\rangle$. Conversely, $ee'' = e^2 + ee' - e^2e' = e + ee' - ee' = e$. By symmetry, $e'e'' = e'$. So $\langle e, e'\rangle \subset \langle e''\rangle$ and $e''^2 = ee'' + e'e'' - ee'e'' = e''$. Thus (4) holds.

For (4), induct on r. Thus (3) yields (4).

For (5), recall that every element of R is idempotent. Thus (4) yields (5). □

2. Prime Ideals

Exercise **(2.2)**. — Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals, and $\mathfrak{p}$ a prime ideal. Prove that these conditions are equivalent: (1) $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$; and (2) $\mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p}$; and (3) $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$.

Solution: Trivially, (1) implies (2). If (2) holds, then (3) follows as $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$. Finally, assume $\mathfrak{a} \not\subset \mathfrak{p}$ and $\mathfrak{b} \not\subset \mathfrak{p}$. Then there are $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$ with $x, y \notin \mathfrak{p}$. Hence, since $\mathfrak{p}$ is prime, $xy \notin \mathfrak{p}$. However, $xy \in \mathfrak{a}\mathfrak{b}$. Thus (3) implies (1). □

Exercise **(2.4)**. — Given a prime number p and an integer $n \geq 2$, prove that the residue ring $\mathbb{Z}/\langle p^n\rangle$ does not contain a domain as a subring.

Solution: Any subring of $\mathbb{Z}/\langle p^n\rangle$ must contain 1, and 1 generates $\mathbb{Z}/\langle p^n\rangle$ as an abelian group. So $\mathbb{Z}/\langle p^n\rangle$ contains no proper subrings. However, $\mathbb{Z}/\langle p^n\rangle$ is not a domain, because in it, $p \cdot p^{n-1} = 0$ but neither p nor p^{n-1} is 0. □

(1) The complement of a multiplicative subset is a prime ideal.
(2) Given two prime ideals, their intersection is prime.
(3) Given two prime ideals, their sum is prime.
(4) Given a ring map $\varphi\colon R \to R'$, the operation φ^{-1} carries maximal ideals of R' to maximal ideals of R.
(5) In (**1.9**), an ideal $n' \subset R/\mathfrak{a}$ is maximal if and only if $\kappa^{-1}\mathfrak{n}' \subset R$ is maximal.

Solution: (1) False. In the ring $\mathbb{Z}$, consider the set S of powers of 2. The complement T of S contains 3 and 5, but not 8; so T is not an ideal.

(2) False. In the ring $\mathbb{Z}$, consider the prime ideals $\langle 2\rangle$ and $\langle 3\rangle$; their intersection $\langle 2\rangle \cap \langle 3\rangle$ is equal to $\langle 6\rangle$, which is not prime.

(3) False. Since $2 \cdot 3 - 5 = 1$, we have $\langle 3\rangle + \langle 5\rangle = \mathbb{Z}$.

(4) False. Let $\varphi\colon \mathbb{Z} \to \mathbb{Q}$ be the inclusion map. Then $\varphi^{-1}\langle 0\rangle = \langle 0\rangle$.

(5) True. By(**1.9**), the operation $\mathfrak{b}' \mapsto \kappa^{-1}\mathfrak{b}'$ sets up an inclusion-preserving bijective correspondence between the ideals $\mathfrak{b}' \supset \mathfrak{n}'$ and the ideals $\mathfrak{b} \supset \kappa^{-1}\mathfrak{n}'$. □

Exercise (**2.23**). — Let k be a field, $P := k[X_1, \dots, X_n]$ the polynomial ring, $f \in P$ nonzero. Let d be the highest power of any variable appearing in f.

(1) Let $S \subset k$ have at least $d+1$ elements. Proceeding by induction on n, find $a_1, \dots, a_n \in S$ with $f(a_1, \dots, a_n) \neq 0$.

(2) Using the algebraic closure K of k, find a maximal ideal $\mathfrak{m}$ of P with $f \notin \mathfrak{m}$.

Solution: Consider (1). Assume $n = 1$. Then f has at most d roots by [2, (1.8), p. 392]. So $f(a_1) \neq 0$ for some $a_1 \in S$.

Assume $n > 1$. Say $f = \sum_j g_j X_1^j$ with $g_j \in k[X_2, \dots, X_n]$. But $f \neq 0$. So $g_i \neq 0$ for some i. By induction, $g_i(a_2, \dots, a_n) \neq 0$ for some $a_2, \dots, a_n \in S$. By $n = 1$, find $a_1 \in S$ such that $f(a_1, \dots, a_n) = \sum_j g_j(a_2, \dots, a_n)a_1^j \neq 0$. Thus (1) holds.

Consider (2). As K is infinite, (1) yields $a_1, \dots, a_n \in K$ with $f_i(a_1, \dots, a_n) \neq 0$. Define $\varphi\colon P \to K$ by $\varphi(X_i) = a_i$. Then $\mathrm{Im}(\varphi) \subset K$ is the k-subalgebra generated by the a_i. It is a field by [2, (2.6), p. 495]. Set $\mathfrak{m} := \mathrm{Ker}(\varphi)$. Then $\mathfrak{m}$ is maximal by (**1.6.1**) and (**2.17**), and $f_i \notin \mathfrak{m}$ as $\varphi(f_i) = f_i(a_1, \dots, a_n) \neq 0$. Thus (2) holds. □

Exercise (**2.26**). — Prove that, in a PID, elements x and y are **relatively prime** (share no prime factor) if and only if the ideals $\langle x\rangle$ and $\langle y\rangle$ are comaximal.

Solution: Say $\langle x\rangle + \langle y\rangle = \langle d\rangle$. Then $d = \gcd(x, y)$, as is easy to check. The assertion is now obvious. □

Exercise (**2.29**). — Preserve the setup of (**2.28**). Let $f := a_0X^n + \cdots + a_n$ be a polynomial of positive degree n. Assume that R has infinitely many prime elements p, or simply that there is a p such that $p \nmid a_0$. Show that $\langle f\rangle$ is not maximal.

Solution: Set $\mathfrak{a} := \langle p, f\rangle$. Then $\mathfrak{a} \supsetneqq \langle f\rangle$, because p is not a multiple of f. Set $k := R/\langle p\rangle$. Since p is irreducible, k is a domain by (**2.6**) and (**2.8**). Let $f' \in k[X]$ denote the image of f. By hypothesis, $\deg(f') = n \geq 1$. Hence f' is not a unit by (**2.3**) since k is a domain. Therefore, $\langle f'\rangle$ is proper. But $P/\mathfrak{a} \xrightarrow{\sim} k[X]/\langle f'\rangle$ by (**1.7**) and (**1.9**). So $\mathfrak{a}$ is proper. Thus $\langle f\rangle$ is not maximal. □

3. Radicals

Exercise (**3.3**). — Let R be a ring, $\mathfrak{a} \subset \mathrm{rad}(R)$ an ideal, $w \in R$, and $w' \in R/\mathfrak{a}$ its residue. Prove that $w \in R^\times$ if and only if $w' \in (R/\mathfrak{a})^\times$. What if $\mathfrak{a} \not\subset \mathrm{rad}(R)$?

Solution: First, assume S is saturated multiplicative. Take $x \in R - S$. Then $xy \notin S$ for all $y \in R$; in other words, $\langle x \rangle \cap S = \emptyset$. Then (3.12) gives a prime $\mathfrak{p} \supset \langle x \rangle$ with $\mathfrak{p} \cap S = \emptyset$. Thus $R - S$ is a union of primes.

Conversely, assume $R - S$ is a union of primes $\mathfrak{p}$. Then $1 \in S$ as 1 lies in no $\mathfrak{p}$. Take $x, y \in R$. Then $x, y \in S$ if and only if x, y lie in no $\mathfrak{p}$; if and only if xy lies in no $\mathfrak{p}$, as every $\mathfrak{p}$ is prime; if and only if $xy \in S$. Thus S is saturated multiplicative. □

Exercise **(3.17)**. — Let R be a ring, and S a multiplicative subset. Define its **saturation** to be the subset

$$\overline{S} := \{\, x \in R \mid \text{there is } y \in R \text{ with } xy \in S \,\}.$$

(1) Show (a) that $\overline{S} \supset S$, and (b) that $\overline{S}$ is saturated multiplicative, and (c) that any saturated multiplicative subset T containing S also contains $\overline{S}$.

(2) Show that $R - \overline{S}$ is the union U of all the primes $\mathfrak{p}$ with $\mathfrak{p} \cap S = \emptyset$.

(3) Let $\mathfrak{a}$ be an ideal; assume $S = 1 + \mathfrak{a}$; set $W := \bigcup_{\mathfrak{p} \supset \mathfrak{a}} \mathfrak{p}$. Show $R - \overline{S} = W$.

(4) Given $f \in R$, let $\overline{S}_f$ denote the saturation of the multiplicative subset of all powers of f. Given $f, g \in R$, show $\overline{S}_f \subset \overline{S}_g$ if and only if $\sqrt{\langle f \rangle} \supset \sqrt{\langle g \rangle}$.

Solution: Consider (1). Trivially, if $x \in S$, then $x \cdot 1 \in S$. Thus (a) holds.

Hence $1 \in \overline{S}$ as $1 \in S$. Now, take $x, x' \in \overline{S}$. Then there are $y, y' \in R$ with $xy,\ x'y' \in S$. But S is multiplicative. So $(xx'')(yy') \in S$. Hence $xx' \in \overline{S}$. Thus $\overline{S}$ is multiplicative. Further, take $x, x' \in R$ with $xx' \in \overline{S}$. Then there is $y \in R$ with $xx'y \in S$. So $x, x' \in \overline{S}$. Thus $\overline{S}$ is saturated. Thus (b) holds.

Finally, consider (c). Given $x \in \overline{S}$, there is $y \in R$ with $xy \in S$. So $xy \in T$. But T is saturated multiplicative. So $x \in T$. Thus $T \supset \overline{S}$. Thus (c) holds.

Consider (2). Plainly, $R - U$ contains S. Further, $R - U$ is saturated multiplicative by (3.16). So $R - U \supset \overline{S}$ by (1)(c). Thus $U \subset R - \overline{S}$. Conversely, $R - \overline{S}$ is a union of primes $\mathfrak{p}$ by (3.16). Plainly, $\mathfrak{p} \cap S = \emptyset$ for all $\mathfrak{p}$. So $U \supset R - \overline{S}$. Thus (2) holds.

For (3), first take a prime $\mathfrak{p}$ with $\mathfrak{p} \cap S = \emptyset$. Then $1 \notin \mathfrak{p} + \mathfrak{a}$; else, $1 = p + a$ with $p \in \mathfrak{p}$ and $a \in \mathfrak{a}$, and so $1 - p = a \in \mathfrak{p} \cap S$. So $\mathfrak{p} + \mathfrak{a}$ lies in a maximal ideal $\mathfrak{m}$ by (3.12). Then $\mathfrak{a} \subset \mathfrak{m}$; so $\mathfrak{m} \subset W$. But also $\mathfrak{p} \subset \mathfrak{m}$. Thus $U \subset W$.

Conversely, take $\mathfrak{p} \supset \mathfrak{a}$. Then $1 + \mathfrak{p} \subset 1 + \mathfrak{a} = S$. But $\mathfrak{p} \cap (1 + \mathfrak{p}) = \emptyset$. So $\mathfrak{p} \cap S = \emptyset$. Thus $U \supset W$. Thus $U = W$. Thus (2) yields (3).

Consider (4). By (1), $\overline{S}_f \subset \overline{S}_g$ if and only if $f \in \overline{S}_g$. By definition of saturation, $f \in \overline{S}_g$ if and only if $hf = g^n$ for some h and n. By definition of radical, $hf = g^n$ for some h and n if and only if $g \in \sqrt{\langle f \rangle}$. Plainly, $g \in \sqrt{\langle f \rangle}$ if and only if $\sqrt{\langle g \rangle} \subset \sqrt{\langle f \rangle}$. Thus (4) holds. □

Exercise **(3.18)**. — Let R be a nonzero ring, S a subset. Show S is maximal in the set $\mathfrak{S}$ of multiplicative subsets T of R with $0 \notin T$ if and only if $R - S$ is a **minimal** prime — that is, it is a prime containing no smaller prime.

Solution: First, assume S is maximal in $\mathfrak{S}$. Then S is equal to its saturation $\overline{S}$, as $S \subset \overline{S}$ and $\overline{S}$ is multiplicative by (3.17) (1) (a), (b) and as $0 \in \overline{S}$ would imply $0 = 0 \cdot y \in S$ for some y. So $R - S$ is a union of primes $\mathfrak{p}$ by (3.16). Fix a $\mathfrak{p}$. Then (3.14) yields in $\mathfrak{p}$ a minimal prime $\mathfrak{q}$. Then $S \subset R - \mathfrak{q}$. But $R - \mathfrak{q} \in \mathfrak{S}$ by (2.1). As S is maximal, $S = R - \mathfrak{q}$, or $R - S = \mathfrak{q}$. Thus $R - S$ is a minimal prime.

Conversely, assume $R - S$ is a minimal prime $\mathfrak{q}$. Then $S \in \mathfrak{S}$ by (2.1). Given $T \in \mathfrak{S}$ with $S \subset T$, note $R - \overline{T} = \bigcup \mathfrak{p}$ with $\mathfrak{p}$ prime by (3.16). Fix a $\mathfrak{p}$. Now, $S \subset T \subset \overline{T}$. So $\mathfrak{q} \supset \mathfrak{p}$. But $\mathfrak{q}$ is minimal. So $\mathfrak{q} = \mathfrak{p}$. But $\mathfrak{p}$ is arbitrary, and

$\bigcup \mathfrak{p} = R - \overline{T}$. Hence $\mathfrak{q} = R - \overline{T}$. So $S = \overline{T}$. Hence $S = T$. Thus S is maximal. □

Exercise **(3.20)**. — Let k be a field, $\mathcal{S} \subset k$ a subset of cardinality d at least 2.

(1) Let $P := k[X_1, \dots, X_n]$ be the polynomial ring, $f \in P$ nonzero. Assume the highest power of any X_i in f is less than d. Proceeding by induction on n, show there are $a_1, \dots, a_n \in \mathcal{S}$ with $f(a_1, \dots, a_n) \neq 0$.

(2) Let V be a k-vector space, and $W_1, \dots, W_r$ proper subspaces. Assume $r < d$. Show $\bigcup_i W_i \neq V$.

(3) In (2), let $W \subset \bigcup_i W_i$ be a subspace. Show $W \subset W_i$ for some i.

(4) Let R a k-algebra, $\mathfrak{a}, \mathfrak{a}_1, \dots, \mathfrak{a}_r$ ideals with $\mathfrak{a} \subset \bigcup_i \mathfrak{a}_i$. Show $\mathfrak{a} \subset \mathfrak{a}_i$ for some i.

Solution: For (1), first assume $n = 1$. Then f has degree at most d, so at most d roots by [2, (1.8), p. 392]. So there's $a_1 \in \mathcal{S}$ with $f(a_1) \neq 0$.

Assume $n > 1$. Say $f = \sum_j g_j X_1^j$ with $g_j \in k[X_2, \dots, X_n]$. But $f \neq 0$. So $g_i \neq 0$ for some i. By induction, there are $a_2, \dots, a_n \in \mathcal{S}$ with $g_i(a_2, \dots, a_n) \neq 0$. So there's $a_1 \in \mathcal{S}$ with $f(a_1, \dots, a_n) = \sum_j g_j(a_2, \dots, a_n) a_1^j \neq 0$. Thus (1) holds.

For (2), for all i, take $v_i \in V - W_i$. Form their span $V' \subset V$. Set $n := \dim V'$ and $W_i' := W_i \cap V'$. Then $n < \infty$, and it suffices to show $\bigcup_i W_i' \neq V'$.

Identify V' with k^n. Form the polynomial ring $P := k[X_1, \dots, X_n]$. For each i, take a linear form $f_i \in P$ that vanishes on W_i'. Set $f := f_1 \cdots f_r$. Then r is the highest power of any variable in f. But $r < d$. So (1) yields $a_1, \dots, a_n \in \mathcal{S}$ with $f(a_1, \dots, a_n) \neq 0$. Then $(a_1, \dots, a_n) \in V' - \bigcup_i W_i'$.

For (3), for all i, set $U_i := W \cap W_i$. Then $\bigcup_i U_i = W$. So (2) implies $U_i = W$ for some i. Thus $W \subset W_i$.

Finally, (4) is a special case of (3), as every ideal is a k-vector space. □

Exercise **(3.21)**. — Let k be a field, $R := k[X, Y]$ the polynomial ring in two variables, $\mathfrak{m} := \langle X, Y \rangle$. Show $\mathfrak{m}$ is a union of strictly smaller primes.

Solution: Since R is a UFD, and $\mathfrak{m}$ is maximal, so prime, any nonzero $f \in \mathfrak{m}$ has a prime factor $p \in \mathfrak{m}$. Thus $\mathfrak{m} = \bigcup_p \langle p \rangle$, but $\mathfrak{m} \neq \langle p \rangle$ as $\mathfrak{m}$ is not principal. □

Exercise **(3.23)**. — Find the nilpotents in $\mathbb{Z}/\langle n \rangle$. In particular, take $n = 12$.

Solution: An integer m is nilpotent modulo n if and only if some power m^k is divisible by n. The latter holds if and only if every prime factor of n occurs in m. In particular, in $\mathbb{Z}/\langle 12 \rangle$, the nilpotents are 0 and 6. □

Exercise **(3.24)**. — Let R be a ring. (1) Assume every ideal not contained in $\operatorname{nil}(R)$ contains a nonzero idempotent. Prove that $\operatorname{nil}(R) = \operatorname{rad}(R)$. (2) Assume R is Boolean. Prove that $\operatorname{nil}(R) = \operatorname{rad}(R) = \langle 0 \rangle$.

Solution: or (1), recall **(3.22.1)**, that $\operatorname{nil}(R) \subset \operatorname{rad}(R)$. To prove the opposite inclusion, set $R' := R/\operatorname{nil}(R)$. Assume $\operatorname{rad}(R') \neq \langle 0 \rangle$. Then there is a nonzero idempotent $e \in \operatorname{rad}(R')$. Then $e(1-e) = 0$. But $1 - e$ is a unit by **(3.2)**. So $e = 0$, a contradiction. Hence $\operatorname{rad}(R') = \langle 0 \rangle$. Thus **(1.9)** yields (1).

For (2), recall from **(1.2)** that every element of R is idempotent. So $\operatorname{nil}(R) = \langle 0 \rangle$, and every nonzero ideal contains a nonzero idempotent. Thus (1) yields (2). □

Exercise **(3.25)**. — Let $\varphi\colon R \to R'$ be a ring map, $\mathfrak{b} \subset R'$ a subset. Prove

$$\varphi^{-1}\sqrt{\mathfrak{b}} = \sqrt{\varphi^{-1}\mathfrak{b}}.$$

$R := P/\langle X_2^2, X_3^3, \dots\rangle$. Let a_n be the residue of X_n. Then $a_n^n = 0$, but $\sum a_n X^n$ is not nilpotent. Thus (1) holds.

For (2), given $g = \sum b_n X^n \in \operatorname{rad}(R[[X]])$, note that $1 + fg$ is a unit if and only if $1 + a_0 b_0$ is a unit by **(3.10)**. Thus **(3.2)** yields (2) holds.

For (3), note $\mathfrak{M}$ contains X and $\mathfrak{m}$, so the ideal they generate. But $f = a_0 + Xg$ for some $g \in R[[X]]$. So if $f \in \mathfrak{M}$, then $a_0 \in \mathfrak{M} \cap R = \mathfrak{m}$. Thus (3) holds.

For (4), note that $X \in \operatorname{rad}(R[[X]])$ by (2). So X and $\mathfrak{m}$ generate $\mathfrak{M}$ by (3). So $P/\mathfrak{n} = R/\mathfrak{m}$ by **(3.10)**. Thus **(2.17)** yields (4).

In (5), plainly $\mathfrak{a}R[[X]] \subset \mathfrak{A}$. Now, assume $f := \sum a_n X^n \in \mathfrak{A}$, or all $a_n \in \mathfrak{a}$. Say $b_1, \dots, b_m \in \mathfrak{a}$ generate. Then $a_n = \sum_{i=1}^m c_{ni} b_i$ for some $c_{ni} \in R$. Thus, as desired,

$$f = \sum_{n\ge 0}\Big(\sum_{i=1}^m c_{ni} b_i\Big) X^n = \sum_{i=1}^m b_i \Big(\sum_{n\ge 0} c_{ni} X^n\Big) \in \mathfrak{a}R[[X]].$$

For a counterexample, take $a_0, a_1, \dots$ to be variables. Take $R := \mathbb{Z}[a_1, a_2, \dots]$ and $\mathfrak{a} := \langle a_1, a_2, \dots\rangle$. Given $g \in \mathfrak{a}R[[X]]$, say $g = \sum_{i=1}^m b_i g_i$ with $b_i \in \mathfrak{a}$ and $g_i = \sum_{n\ge 0} b_{in} X^n$. Choose p greater than the maximum n such that a_n occurs in any b_i. Then $\sum_{i=1}^m b_i b_{in} \in \langle a_1, \dots, a_{p-1}\rangle$, but $a_p \notin \langle a_1, \dots, a_{p-1}\rangle$. Therefore, $g \ne f := \sum a_n X^n$. Thus $f \notin \mathfrak{a}R[[X]]$, but $f \in \mathfrak{A}$. □

4. Modules

Exercise **(4.3)**. — Let R be a ring, M a module. Consider the set map
$$\rho\colon \operatorname{Hom}(R, M) \to M \quad \text{defined by} \quad \rho(\theta) := \theta(1).$$
Show that ρ is an isomorphism, and describe its inverse.

Solution: First off, ρ is R-linear, because
$$\rho(x\theta + x'\theta') = (x\theta + x'\theta')(1) = x\theta(1) + x'\theta'(1) = x\rho(\theta) + x'\rho(\theta').$$

Set $H := \operatorname{Hom}(R, M)$. Define $\alpha\colon M \to H$ by $\alpha(m)(x) := xm$. It is easy to check that $\alpha\rho = 1_H$ and $\rho\alpha = 1_M$. Thus ρ and α are inverse isomorphisms by **(4.2)**. □

Exercise **(4.12)**. — Let R be a domain, and $x \in R$ nonzero. Let M be the submodule of $\operatorname{Frac}(R)$ generated by 1, x^{-1}, $x^{-2}, \dots$. Suppose that M is finitely generated. Prove that $x^{-1} \in R$, and conclude that $M = R$.

Solution: Suppose M is generated by $m_1, \dots, m_k$. Say $m_i = \sum_{j=0}^{n_i} a_{ij} x^{-j}$ for some n_i and $a_{ij} \in R$. Set $n := \max\{n_i\}$. Then 1, $x^{-1}, \dots, x^{-n}$ generate M. So
$$x^{-(n+1)} = a_n x^{-n} + \cdots + a_1 x^{-1} + a_0$$
for some $a_i \in R$. Thus
$$x^{-1} = a_n + \cdots + a_1 x^{n-1} + a_0 x^n \in R.$$

Finally, as $x^{-1} \in R$ and R is a ring, also 1, x^{-1}, $x^{-2}, \dots \in R$; so $M \subset R$. Conversely, $M \supset R$ as $1 \in M$. Thus $M = R$. □

Exercise **(4.13)**. — A finitely generated free module has finite rank.

Solution: Say e_λ for $\lambda \in \Lambda$ form a free basis, and $m_1, \dots, m_r$ generate. Then $m_i = \sum x_{ij} e_{\lambda_j}$ for some x_{ij}. Consider the e_{λ_j} that occur. Plainly, they are finite in number, and generate. So they form a finite free basis, as desired. □

Exercise **(4.16)**. — Let Λ be an infinite set, R_λ a nonzero ring for $\lambda \in \Lambda$. Endow $\prod R_\lambda$ and $\bigoplus R_\lambda$ with componentwise addition and multiplication. Show that $\prod R_\lambda$ has a multiplicative identity (so is a ring), but that $\bigoplus R_\lambda$ does not (so is not a ring).

Solution: Consider the vector (1) whose every component is 1. Obviously, (1) is a multiplicative identity of $\prod R_\lambda$. On the other hand, no restricted vector (x_λ) can be a multiplicative identity in $\bigoplus R_\lambda$; indeed, because Λ is infinite, x_μ must be zero for some μ. So $(x_\lambda)\cdot(y_\lambda) \neq (y_\lambda)$ if $y_\mu \neq 0$. □

Exercise **(4.17)**. — Let R be a ring, M a module, and M', M'' submodules. Show that $M = M' \oplus M''$ if and only if $M = M' + M''$ and $M' \cap M'' = 0$.

Solution: Assume $M = M' \oplus M''$. Then M is the set of pairs (m', m'') with $m' \in M'$ and $m'' \in M''$ by **(4.15)**; further, M' is the set of $(m', 0)$, and M' is that of $(0, m'')$. So plainly $M = M' + M''$ and $M' \cap M'' = 0$.

Conversely, consider the map $M' \oplus M'' \to M$ given by $(m', m'') \mapsto m' + m''$. It is surjective if $M = M' + M''$. It is injective if $M' \cap M'' = 0$; indeed, if $m' + m'' = 0$, then $m' = -m'' \in M' \cap M'' = 0$, and so $(m', m'') = 0$ as desired. □

Exercise **(4.18)**. — Let L, M, and N be modules. Consider a diagram

$$L \underset{\rho}{\overset{\alpha}{\rightleftarrows}} M \underset{\sigma}{\overset{\beta}{\rightleftarrows}} N$$

where α, β, ρ, and σ are homomorphisms. Prove that

$$M = L \oplus N \quad \text{and} \quad \alpha = \iota_L,\ \beta = \pi_N,\ \sigma = \iota_N,\ \rho = \pi_L$$

if and only if the following relations hold:

$$\beta\alpha = 0,\ \beta\sigma = 1,\ \rho\sigma = 0,\ \rho\alpha = 1, \text{ and } \alpha\rho + \sigma\beta = 1.$$

Solution: If $M = L \oplus N$ and $\alpha = \iota_L$, $\beta = \pi_N$, $\sigma\iota_N$, $\rho = \pi_L$, then the definitions immediately yield $\alpha\rho + \sigma\beta = 1$ and $\beta\alpha = 0$, $\beta\sigma = 1$, $\rho\sigma = 0$, $\rho\alpha = 1$.

Conversely, assume $\alpha\rho + \sigma\beta = 1$ and $\beta\alpha = 0$, $\beta\sigma = 1$, $\rho\sigma = 0$, $\rho\alpha = 1$. Consider the maps $\varphi\colon M \to L \oplus N$ and $\theta\colon L \oplus N \to M$ given by $\varphi m := (\rho m,\ \beta m)$ and $\theta(l, n) := \alpha l + \sigma n$. They are inverse isomorphisms, because

$$\varphi\theta(l, n) = (\rho\alpha l + \rho\sigma n,\ \beta\alpha l + \beta\sigma n) = (l, n) \quad \text{and} \quad \theta\varphi m = \alpha\rho m + \sigma\beta m = m.$$

Lastly, $\beta = \pi_N\varphi$ and $\rho = \pi_L\varphi$ by definition of φ, and $\alpha = \theta\iota_L$ and $\sigma = \theta\iota_N$ by definition of θ. □

Exercise **(4.19)**. — Let L be a module, Λ a nonempty set, M_λ a module for $\lambda \in \Lambda$. Prove that the injections $\iota_\kappa\colon M_\kappa \to \bigoplus M_\lambda$ induce an injection

$$\bigoplus \mathrm{Hom}(L, M_\lambda) \hookrightarrow \mathrm{Hom}(L, \bigoplus M_\lambda),$$

and that it is an isomorphism if L is finitely generated.

Solution: For $\lambda \in \Lambda$, let $\alpha_\lambda\colon L \to M_\lambda$ be maps, almost all 0. Then

$$\big(\sum \iota_\lambda\alpha_\lambda\big)(l) = \big(\alpha_\lambda(l)\big) \in \bigoplus M_\lambda.$$

So if $\sum \iota_\lambda\alpha_\lambda = 0$, then $\alpha_\lambda = 0$ for all λ. Thus the ι_κ induce an injection.

Assume L is finitely generated, say by $l_1, \ldots, l_k$. Let $\alpha\colon L \to \bigoplus M_\lambda$ be a map. Then each $\alpha(l_i)$ lies in a finite direct subsum of $\bigoplus M_\lambda$. So $\alpha(L)$ lies in one too. Set $\alpha_\kappa := \pi_\kappa\alpha$ for all $\kappa \in \Lambda$. Then almost all α_κ vanish. So (α_κ) lies in $\bigoplus \mathrm{Hom}(L, M_\lambda)$, and $\sum \iota_\kappa\alpha_\kappa = \alpha$. Thus the ι_κ induce a surjection, so an isomorphism. □

quotient of $\mathrm{Hom}(R^{\oplus m}, N)$ by **(5.9)**. So $\mathrm{Hom}(P, N)$ is finitely generated too.

Suppose now there is a finite presentation $F_2 \to F_1 \to N \to 0$. Then **(5.22)** and **(5.23)** yield the exact sequence

$$\mathrm{Hom}(R^{\oplus m}, F_2) \to \mathrm{Hom}(R^{\oplus m}, F_1) \to \mathrm{Hom}(R^{\oplus m}, N) \to 0.$$

But the $\mathrm{Hom}(R^{\oplus m}, F_i)$ are free of finite rank by **(4.15.1)** and **(4.15.2)**. Thus $\mathrm{Hom}(R^{\oplus m}, N)$ is finitely presented.

As above, $\mathrm{Hom}(K, N)$ is finitely generated. Consider the (split) exact sequence

$$0 \to \mathrm{Hom}(K, N) \to \mathrm{Hom}(R^{\oplus m}, N) \to \mathrm{Hom}(P, N) \to 0.$$

Thus **(5.28)** implies $\mathrm{Hom}(P, N)$ is finitely presented. □

Exercise **(5.26)**. — Let R be a ring, and $0 \to L \to R^n \to M \to 0$ an exact sequence. Prove M is finitely presented if and only if L is finitely generated.

Solution: Assume M is finitely presented; say $R^l \to R^m \to M \to 0$ is a finite presentation. Let L' be the image of R^l. Then $L' \oplus R^n \simeq L \oplus R^m$ by Schanuel's Lemma **(5.25)**. Hence L is a quotient of $R^l \oplus R^n$. Thus L is finitely generated.

Conversely, assume L is generated by ℓ elements. They yield a surjection $R^\ell \twoheadrightarrow L$ by **(4.10)**(1). It yields a sequence $R^\ell \to R^n \to M \to 0$. The latter is plainly exact. Thus M is finitely presented. □

Exercise **(5.27)**. — Let R be a ring, $X_1, X_2, \ldots$ infinitely many variables. Set $P := R[X_1, X_2, \ldots]$ and $M := P/\langle X_1, X_2, \ldots\rangle$. Is M finitely presented? Explain.

Solution: No, otherwise by **(5.26)**, the ideal $\langle X_1, X_2, \ldots\rangle$ would be generated by some $f_1, \ldots, f_n \in P$, so also by $X_1, \ldots, X_m$ for some m, but plainly it isn't. □

Exercise **(5.29)**. — Let $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ be a short exact sequence with M finitely generated and N finitely presented. Prove L is finitely generated.

Solution: Let R be the ground ring. Say M is generated by m elements. They yield a surjection $\mu\colon R^m \twoheadrightarrow M$ by **(4.10)**(1). As in **(5.28)**, μ induces the following commutative diagram, with λ surjective:

$$\begin{array}{ccccccccc}
0 & \to & K & \to & R^m & \to & N & \to & 0 \\
 & & \lambda\big\downarrow & & \mu\big\downarrow & & 1_N\big\downarrow & & \\
0 & \to & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \to & 0
\end{array}$$

By **(5.26)**, K is finitely generated. Thus L is too, as λ is surjective. □

Exercise **(5.36)**. — Let R be a ring, and $a_1, \ldots, a_m \in R$ with $\langle a_1\rangle \supset \cdots \supset \langle a_m\rangle$. Set $M := (R/\langle a_1\rangle) \oplus \cdots \oplus (R/\langle a_m\rangle)$. Show that $F_r(M) = \langle a_1 \cdots a_{m-r}\rangle$.

Solution: Form the presentation $R^m \xrightarrow{\alpha} R^m \to M \to 0$ where α has matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_m \end{pmatrix}$$

Set $s := m - r$. Now, $a_i \in \langle a_{i-1}\rangle$ for all $i > 1$. Hence $a_{i_1} \cdots a_{i_s} \in \langle a_1 \cdots a_s\rangle$ for all $1 \le i_1 < \cdots < i_s \le m$. Thus $I_s(\mathbf{A}) = \langle a_1 \cdots a_s\rangle$, as desired. □

Solution: The monomials form a free basis, so P is faithfully flat by (**9.7**). □

Exercise (**9.10**). — Let R be a ring, M and N flat modules. Show that $M \otimes N$ is flat. What if "flat" is replaced everywhere by "faithfully flat"?

Solution: Associativity (**8.10**) yields $(M \otimes N) \otimes \bullet = M \otimes (N \otimes \bullet)$; in other words, $(M \otimes N) \otimes \bullet = (M \otimes \bullet) \circ (N \otimes \bullet)$. So $(M \otimes N) \otimes \bullet$ is the composition of two exact functors. Hence it is exact. Thus $M \otimes N$ is flat.

Similarly if M and N are faithfully flat, then $M \otimes N \otimes \bullet$ is faithful and exact. So $M \otimes N$ is faithfully flat. □

Exercise (**9.11**). — Let R be a ring, M a flat module, R' an algebra. Show that $M \otimes_R R'$ is flat over R'. What if "flat" is replaced everywhere by "faithfully flat"?

Solution: Cancellation (**8.11**) yields $(M \otimes_R R') \otimes_{R'} \bullet = M \otimes_R \bullet$. But $M \otimes_R \bullet$ is exact, as M is flat over R. Thus $M \otimes_R R'$ is flat over R'.

Similarly, if M is faithfully flat over R, then $M \otimes_R \bullet$ is faithful too. Thus $M \otimes_R R'$ is faithfully flat over R'. □

Exercise (**9.12**). — Let R be a ring, R' a flat algebra, M a flat R'-module. Show that M is flat over R. What if "flat" is replaced everywhere by "faithfully flat"?

Solution: Cancellation (**8.11**) yields $M \otimes_R \bullet = M \otimes_{R'} (R' \otimes_R \bullet)$. But $R' \otimes_R \bullet$ and $M \otimes_{R'} \bullet$ are exact; so their composition $M \otimes_R \bullet$ is too. Thus M is flat over R.

Similarly, as the composition of two faithful functors is, plainly, faithful, the assertion remains true if "flat" is replaced everywhere by "faithfully flat." □

Exercise (**9.13**). — Let R be a ring, R' an algebra, R'' an R'-algebra, and M an R''-module. Assume that M is flat over R and faithfully flat over R'. Prove that R' is flat over R.

Solution: Let $N' \to N$ be an injective map of R-modules. Then the map $N' \otimes_R M \to N \otimes_R M$ is injective as M is flat over R. But by Cancellation (**8.11**), that map is equal to this one:

$$(N' \otimes_R R') \otimes_{R'} M \to (N \otimes_R R') \otimes_{R'} M.$$

And M is faithfully flat over R'. Hence the map $N' \otimes_R R' \to N \otimes_R R'$ is injective by (**9.4**). Thus R' is flat over R. □

Exercise (**9.14**). — Let R be a ring, $\mathfrak{a}$ an ideal. Assume $R/\mathfrak{a}$ is flat. Show $\mathfrak{a} = \mathfrak{a}^2$.

Solution: Since $R/\mathfrak{a}$ is flat, tensoring it with the inclusion $\mathfrak{a} \hookrightarrow R$ yields an injection $\mathfrak{a} \otimes_R (R/\mathfrak{a}) \hookrightarrow R \otimes_R (R/\mathfrak{a})$. But the image vanishes: $a \otimes r = 1 \otimes ar = 0$. Further, $\mathfrak{a} \otimes_R (R/\mathfrak{a}) = \mathfrak{a}/\mathfrak{a}^2$ by (**8.16**). Hence $\mathfrak{a}/\mathfrak{a}^2 = 0$. Thus $\mathfrak{a} = \mathfrak{a}^2$. □

Exercise (**9.15**). — Let R be a ring, R' a flat algebra. Prove equivalent:

(1) R' is faithfully flat over R.
(2) For every R-module M, the map $M \xrightarrow{\alpha} M \otimes_R R'$ by $\alpha m = m \otimes 1$ is injective.
(3) Every ideal $\mathfrak{a}$ of R is the contraction of its extension, or $\mathfrak{a} = \varphi^{-1}(\mathfrak{a}R')$.
(4) Every prime $\mathfrak{p}$ of R is the contraction of some prime $\mathfrak{q}$ of R', or $\mathfrak{p} = \varphi^{-1}\mathfrak{q}$.
(5) Every maximal ideal $\mathfrak{m}$ of R extends to a proper ideal, or $\mathfrak{m}R' \neq R'$.
(6) Every nonzero R-module M extends to a nonzero module, or $M \otimes_R R' \neq 0$.

$N'_n \subset N_n$. But $N' \supset N$. Thus $N'_n = N_n$ for $n \ge n_1$, as desired.

Let $N'' = \bigoplus N''_n \subset M$ be homogeneous with $N''_n = N_n$ for $n \ge n_2$. Let $m \in N''$ and $p \ge n_2$. Then $R_p m \in \bigoplus_{n \ge n_2} N''_n \subset N$. So $m \in N'$. Thus $N'' \subset N'$. □

Exercise **(20.25)**. — Let R be a graded ring, $\mathfrak{a}$ a homogeneous ideal, and M a graded module. Prove that $\sqrt{\mathfrak{a}}$ and $\mathrm{Ann}(M)$ and $\mathrm{nil}(M)$ are homogeneous.

Solution: Take $x = \sum_{i \ge r}^{r+n} x_i \in R$ with the x_i the homogeneous components.

First, suppose $x \in \sqrt{\mathfrak{a}}$. Say $x^k \in \mathfrak{a}$. Either x_r^k vanishes or it is the initial component of x^k. But $\mathfrak{a}$ is homogeneous. So $x_r^k \in \mathfrak{a}$. So $x_r \in \sqrt{\mathfrak{a}}$. So $x - x_r \in \sqrt{\mathfrak{a}}$ by **(3.31)**. So all the x_i are in $\sqrt{\mathfrak{a}}$ by induction on n. Thus $\sqrt{\mathfrak{a}}$ is homogeneous.

Second, suppose $x \in \mathrm{Ann}(M)$. Let $m \in M$. Then $0 = xm = \sum x_i m$. If m is homogeneous, then $x_i m = 0$ for all i, since M is graded. But M has a set of homogeneous generators. Thus $x_i \in \mathrm{Ann}(M)$ for all i, as desired.

Finally, $\mathrm{nil}(M)$ is homogeneous, as $\mathrm{nil}(M) = \sqrt{\mathrm{Ann}(M)}$ by **(13.28)**. □

Exercise **(20.26)**. — Let R be a Noetherian graded ring, M a finitely generated graded module, Q a submodule. Let $Q^* \subset Q$ be the submodule generated by the homogeneous elements of Q. Assume Q is primary. Then Q^* is primary too.

Solution: Let $x \in R$ and $m \in M$ be homogeneous with $xm \in Q^*$. Assume $x \notin \mathrm{nil}(M/Q^*)$. Then, given $\ell \ge 1$, there is $m' \in M$ with $x^\ell m' \notin Q^*$. So m' has a homogeneous component m'' with $x^\ell m'' \notin Q^*$. Then $x^\ell m'' \notin Q$ by definition of Q^*. Thus $x \notin \mathrm{nil}(M/Q)$. Since Q is primary, $m \in Q$ by **(18.4)**. Since m is homogeneous, $m \in Q^*$. Thus Q^* is primary by **(20.24)**. □

Exercise **(20.30)**. — Under the conditions of **(20.8)**, assume that R is a domain and that its integral closure $\overline{R}$ in $\mathrm{Frac}(R)$ is a finitely generated R-module.

(1) Prove that there is a homogeneous $f \in R$ with $R_f = \overline{R}_f$.

(2) Prove that the Hilbert Polynomials of R and $\overline{R}$ have the same degree and same leading coefficient.

Solution: Let $x_1, \ldots, x_r$ be homogeneous generators of $\overline{R}$ as an R-module. Write $x_i = a_i/b_i$ with $a_i, b_i \in R$ homogeneous. Set $f := \prod b_i$. Then $f x_i \in R$ for each i. So $\overline{R}_f = R_f$. Thus (1) holds.

Consider the short exact sequence $0 \to R \to \overline{R} \to \overline{R}/R \to 0$. Then $(\overline{R}/R)_f = 0$ by **(12.20)**. So $\deg h(\overline{R}/R, n) < \deg h(\overline{R}, n)$ by **(20.10)** and (1). But

$$h(\overline{R}, n) = h(R, n) + h(\overline{R}/R, n)$$

by **(19.9)** and **(20.8)**. Thus (2) holds. □

21. Dimension

Exercise **(21.6)**. — Let A be a Noetherian local ring, N a finitely generated module, $y_1, \ldots, y_r$ a sop for N. Set $N_i := N/\langle y_1, \ldots, y_i \rangle N$. Show $\dim(N_i) = r - i$.

Solution: First, $\dim(N) = r$ by **(21.4)**. Then $\dim(N_i) \ge \dim(N_{i-1}) - 1$ for all i by **(21.5)**, and $\dim(N_r) = 0$ by **(19.18)**. So $\dim(N_i) = r - i$ for all i. □

Exercise **(21.9)**. — Let R be a Noetherian ring, and $\mathfrak{p}$ be a prime minimal containing $x_1, \ldots, x_r$. Given r' with $1 \le r' \le r$, set $R' := R/\langle x_1, \ldots, x_{r'} \rangle$ and $\mathfrak{p}' := \mathfrak{p}/\langle x_1, \ldots, x_{r'} \rangle$. Assume $\mathrm{ht}(\mathfrak{p}) = r$. Prove $\mathrm{ht}(\mathfrak{p}') = r - r'$.

Solution: Assume A is regular. Given a regular sop $x_1, \ldots, x_r$, let's show it's an A-sequence. Set $A_1 := A/\langle x_1 \rangle$. Then A_1 is regular of dimension $r-1$ by (**21.23**). So $x_1 \neq 0$. But A is a domain by (**21.24**). So $x_1 \notin \mathrm{z.div}(A)$. Further, if $r \geq 2$, then the residues of $x_2, \ldots, x_r$ form a regular sop of A_1; so we may assume they form an A_1-sequence by induction on r. Thus $x_1, \ldots, x_r$ is an A-sequence.

Conversely, if $\mathfrak{m}$ is generated by an A-sequence $x_1, \ldots, x_n$, then $n \leq \operatorname{depth}(A) \leq r$ by (**23.4**) and (**23.5**)(3), and $n \geq r$ by (**21.19**). Thus then $n = \operatorname{depth}(A) = r$, and so A is regular and Cohen–Macaulay. □

Exercise (**23.11**). — Let A be a DVR with fraction field K, and $f \in A$ a nonzero nonunit. Prove A is a maximal proper subring of K. Prove $\dim(A) \neq \dim(A_f)$.

Solution: Let R be a ring, $A \subsetneqq R \subset K$. Then there's an $x \in R - A$. Say $x = ut^n$ where $u \in A^\times$ and t is a uniformizing parameter. Then $n < 0$. Set $y := u^{-1}t^{-n-1}$. Then $y \in A$. So $t^{-1} = xy \in R$. Hence $wt^m \in R$ for any $w \in A^\times$ and $m \in \mathbb{Z}$. Thus $R = K$, as desired.

Since f is a nonzero nonunit, $A \subsetneqq A_f \subset K$. Hence $A_f = K$ by the above. So $\dim(A_f) = 0$. But $\dim(A) = 1$ by (**23.10**). □

Exercise (**23.12**). — Let k be a field, $P := k[X, Y]$ the polynomial ring in two variables, $f \in P$ an irreducible polynomial. Say $f = \ell(X, Y) + g(X, Y)$ with $\ell(X, Y) = aX + bY$ for $a, b \in k$ and with $g \in \langle X, Y \rangle^2$. Set $R := P/\langle f \rangle$ and $\mathfrak{p} := \langle X, Y \rangle/\langle f \rangle$. Prove that $R_\mathfrak{p}$ is a DVR if and only if $\ell \neq 0$. (Thus $R_\mathfrak{p}$ is a DVR if and only if the plane curve $C : f = 0$ in k^2 is nonsingular at $(0, 0)$.)

Solution: Set $A := R_\mathfrak{p}$ and $\mathfrak{m} := \mathfrak{p}A$. Then (**12.22**) and (**12.4**) yield

$$A/\mathfrak{m} = (R/\mathfrak{p})_\mathfrak{p} = k \quad \text{and} \quad \mathfrak{m}/\mathfrak{m}^2 = \mathfrak{p}/\mathfrak{p}^2.$$

First, assume $\ell \neq 0$. Now, the k-vector space $\mathfrak{m}/\mathfrak{m}^2$ is generated by the images x and y of X and Y in A. Clearly, the image of f is 0 in $\mathfrak{m}/\mathfrak{m}^2$. Also, $g \in (X, Y)^2$; so its image in $\mathfrak{m}/\mathfrak{m}^2$ is also 0. Hence, the image of ℓ is 0 in $\mathfrak{m}/\mathfrak{m}^2$; that is, x and y are linearly dependent. Now, f cannot generate $\langle X, Y \rangle$, so $\mathfrak{m} \neq 0$; hence, $\mathfrak{m}/\mathfrak{m}^2 \neq 0$ by Nakayama's Lemma, (**10.11**). Therefore, $\mathfrak{m}/\mathfrak{m}^2$ is 1-dimensional over k; hence, $\mathfrak{m}$ is principal by (**10.13**)(2). Now, since f is irreducible, A is a domain. Hence, A is a DVR by (**23.10**).

Conversely, assume $\ell = 0$. Then $f = g \in (X, Y)^2$. So

$$\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{p}/\mathfrak{p}^2 = \langle X, Y \rangle / \langle X, Y \rangle^2.$$

Hence, $\mathfrak{m}/\mathfrak{m}^2$ is 2-dimensional. Therefore, A is not a DVR by (**23.11**). □

Exercise (**23.13**). — Let k be a field, A a ring intermediate between the polynomial ring and the formal power series ring in one variable: $k[X] \subset A \subset k[[X]]$. Suppose that A is local with maximal ideal $\langle X \rangle$. Prove that A is a DVR. (Such local rings arise as rings of power series with curious convergence conditions.)

Solution: Let's show that the ideal $\mathfrak{a} := \bigcap_{n \geq 0} \langle X^n \rangle$ of A is zero. Clearly, $\mathfrak{a}$ is a subset of the corresponding ideal $\bigcap_{n \geq 0} \langle X^n \rangle$ of $k[[X]]$, and the latter ideal is clearly zero. Hence (**23.3**) implies A is a DVR. □

Exercise (**23.14**). — Let L/K be an algebraic extension of fields, $X_1, \ldots, X_n$ variables, P and Q the polynomial rings over K and L in $X_1, \ldots, X_n$.

(1) Let $\mathfrak{q}$ be a prime of Q, and $\mathfrak{p}$ its contraction in P. Prove $\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{q})$.

Ideal Theorem (**21.10**). Then $R[y]_{\mathfrak{p}}$ is Noetherian of dimension 1. But L/K is a finite field extension, so $L/\operatorname{Frac}(R[y])$ is one too. Hence the integral closure R' of $R[y]_{\mathfrak{p}}$ in L is a Dedekind domain by (**26.18**). So by the Going-up Theorem (**14.3**), there's a prime $\mathfrak{q}$ of R' lying over $\mathfrak{p}R[y]_{\mathfrak{p}}$. Then as R' is Dedekind, $R'_{\mathfrak{q}}$ is a DVR of L by (**24.7**). Further, $y \in \mathfrak{q}R'_{\mathfrak{q}}$. Thus $x \notin R'_{\mathfrak{q}}$, as desired. □