$$= 3\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} - 2\begin{vmatrix} 0 & -2 \\ 1 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix}$$

$= 3(4 + 6) – 2(0 + 2) + 1\ (0 – 1)$

$= 30 – 4 – 1$

$|A| \quad = 25$

(b) $\quad |A| = 3\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix}$

$= 3\ (4 + 6) + 1\ (–4 –1)$

$= 30 – 5$

$|A| \quad = 25$

9.7 Properties of the Determinant:

The following properties of determinants are frequently useful in their evaluation:

1. Interchanging the corresponding rows and columns of a determinant does not change its value (i.e., $|A| = |A'|$). For example, consider a determinant

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$= a_1(b_2c_3 – b_3c_2) – b_1(a_2c_3 – a_3c_2)+ c_1(a_2b_3 – a_3b_2)$ ... (2)

Now again consider

$$|B| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Expand it by first column

$|B| \quad = a_1(b_2c_3 – b_3c_2) – b_1(a_2c_3 – a_3c_2) + c_1(a_2b_3 – a_3b_2)$

which is same as equation (2)

so $\quad |B| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

or $\quad |B| = |A|$

2. If two rows or two columns of a determinant are interchanged, the sign of the determinant is changed but its absolute value is unchanged.

For example if

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Consider a determinant, $|B| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$|B| = ka_1(b_2c_3 – b_3c_3) – kb_1(a_2c_3 – a_3c_2) + kc_1(a_2b_3 – a_3b_2)$

$= k(a_1(b_2c_3 – b_3c_3) – b_1(a_2c_3 – a_3c_2) + c_1(a_2b_3 – a_3b_2))$

So $|B| = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Or $|B| = K|A|$

6. The value of a determinant is not changed if each element of any row or of any column is added to (or subtracted from) a constant multiple of the corresponding element of another row or column. For example, if

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Consider a matrix,

$$|B| = \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 & c_1 + kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$= (a_1+ka_2)(b_2c_3–b_3c_2)–(b_1+kb_2)(a_2c_3–a_3c_2)+(c_1+kc_2)(a_2b_3–a_3b_2)$

$= [a_1(b_2c_3 – b_3c_2) –b_1(a_2c_3 – a_3c_2) +c_1(a_2b_3 – a_3b_2)]$

$= [ka_2(b_2c_3 – b_3c_2) –kb_2(a_2c_3 – a_3c_2) +kc_2(a_2b_3 – a_3b_2)]$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

9.8 Solution of Linear Equations by Determinants: (Cramer's Rule)

Consider a system of linear equations in two variables x and y,

$$a_1x + b_1y = c_1 \qquad (1)$$

$$a_2x + b_2y = c_2 \qquad (2)$$

Multiply equation (1) by b_2 and equation (2) by b_1 and subtracting, we get

$$x(a_1b_2 - a_2b_1) = b_2c_1 - b_1c_2$$

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \qquad (3)$$

Again multiply eq. (1) by a_2 and eq. (2) by a_1 and subtracting, we get

$$y(a_2b_1 - a_1b_2) = a_2c_1 - a_1c_2$$

$$y = \frac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2}$$

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \qquad (4)$$

Note that x and y from equations (3) and (4) has the same denominator $a_1b_2 - a_2b_1$. So the system of equations (1) and (2) has solution only when $a_1b_2 - a_2b_1 \neq 0$.

The solutions for x and y of the system of equations (1) and (2) can be written directly in terms of determinants without any algebraic operations, as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

This result is called Cramer's Rule.

Here $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = |A|$ is the determinant of the coefficient of x and y in equations (1) and (2)

If $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = |A|$

and $\quad BA = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Hence AB = BA = I

and $\quad$ therefore $B = A^{-1} = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$

Example 10: Find the inverse, if it exists, of the matrix.

$$A = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$$

Solution:

$|A| = 0 + 2(-2+3) - 3(-2+3) = 2 - 3$

$|A| = -1$, Hence solution exists.

Cofactor of A are:

$A_{11} = 0$, $\quad A_{12} = 1$, $\quad A_{13} = 1$

$A_{21} = 2$, $\quad A_{22} = -3$, $\quad A_{23} = 2$

$A_{31} = 3$, $\quad A_{32} = -3$, $\quad A_{33} = 2$

Matrix of transpose of the cofactors is

$$\text{adj } A = C' = \begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

So

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-1} \begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$$

Q.5 Find the solution set of the following system by means of matrices:

(i) $2x - 3y = -1$, $x + 4y = 5$

(ii) $x + y = 2$, $2x - z = 1$, $2y - 3z = -1$

(iii) $x - 2y + z = -1$, $3x + y - 2z = 4$, $y - z = 1$

(iv) $-4x + 2y - 9z = 2$, $3x + 4y + z = 5$, $x - 3y + 2z = 8$

(v) $x + y - 2z = 3$, $3x - y + z = 0$, $3x + 3y - 6z = 8$

Answers 9.3

Q.1 (i) Non-singular (ii) Singular
(iii) Singular

Q.2 (i) Symmetric (ii) Skew-symmetric
(iii) Symmetric

Q.3 (i) 8 (ii) 5 (iii) 3

Q.4 (i) $\begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{2}{7} & \frac{1}{-7} \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{4}{5} & -\frac{1}{5} & -\frac{4}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{7}{10} \\ \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$

(iv) A^{-1} does not exist.

Q.5 (i) {(1, 1)} (ii) {(1, 1, 1)} (iii) {(1, 1, 0)}
(iv) {(7, -3, -4)} (v) no solution

___11. Value of m for which matrix $\begin{bmatrix} 2 & 3 \\ 6 & m \end{bmatrix}$ is singular.

(a) 6 (b) 3 (c) 8 (d) 9

___12. If $[a_{ij}]$ and $[b_{ij}]$ are of the same order and $a_{ij} = b_{ij}$ then the matrix will be

(a) Singular (b) Null (c) unequal (d) equal

13. Matrix $[a_{ij}]_{mxn}$ is a row matrix if:

(a) $i = 1$ (b) $j = 1$ (c) $m = 1$ (d) $n = 1$

14. Matrix $[c_{ij}]_{mxn}$ is a rectangular if:

(a) $i \neq j$ (b) $i = j$ (c) $m = n$ (d) $m - n \neq 0$

15. If $A = [a_{ij}]_{mxn}$ is a scalar matrix if :

(a) $a_{ij} = 0 \quad \forall\ i \neq j$ (b) $a_{ij} = k\ \forall\ i = j$

(c) $a_{ij} = k \quad \forall\ i \neq j$ (d) (a) and (b)

16. Matrix $A = [a_{ij}]_{mxn}$ is an edentity matrix if :

(a) $\forall\ i = j,\ a_{ij} = 0$ (b) $\forall\ i = j,\ a_{ij} = 1$

(c) $\forall\ i \neq j,\ a_{ij} = 0$ (d) both (b) and (c)

17. Which matrix can be rectangular mayrix ?

(a) Diagonal (b) Identity (c) Scalar (d) None

18. If $A = [a_{ij}]_{mxn}$ then order of kA is :

(a) m x n (b) km x kn (c) km x n (d) m x kn

19. $(A - B)^2 = A^2 - 2AB + B^2$, if and only if :

(a) $A + B = 0$ (b) $AB - BA = 0$ (c) $A^2 + B^2 = 0$ (d) (a) and c

20. If A and B ARE symmetric , then AB =

(a) BA (b) $A^t B^t$ (c) $B^t A^t$ (d) (a) and (c)

Answers

Q.1	(1) c	(2) a	(3) d	(4) a	(5) c	(6) c
	(7) a	(8) b	(9) a	(10) c	(11) d	(12) d
	(13) c	(14) d	(15) d	(16) d	(17) d	(18) a
	(19) b	(20) d				