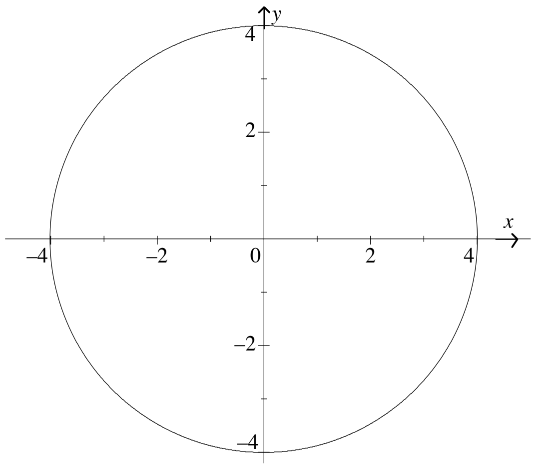


The graph of $x^2 + y^2 = 16$.

Example

Sketch the graph of $f(x) = 3x - x^2$ and find

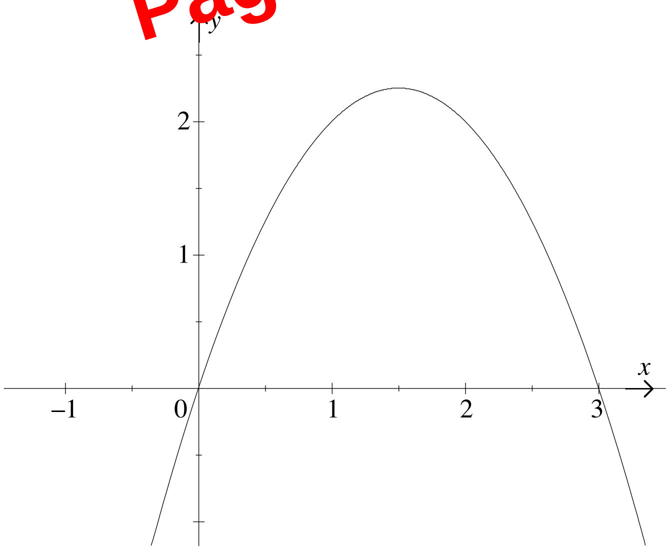
a. the domain and range

b. $f(q)$

c. $f(x^2)$

d. $\frac{f(2+h)-f(2)}{h}$, $h \neq 0$.

Solution



The graph of $f(x) = 3x - x^2$.

a. The domain is all real x. The range is all real y where $y \leq 2.25$.

b. $f(q) = 3q - q^2$

1.3 The absolute value function

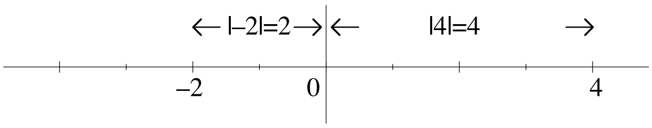
Before we define the absolute value function we will review the definition of the absolute value of a number.

The *Absolute value of a number* x is written $|x|$ and is defined as

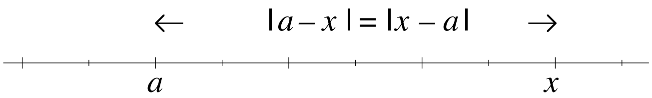
$$|x| = x \text{ if } x \geq 0 \qquad \text{or} \qquad |x| = -x \text{ if } x < 0.$$

That is, $|4| = 4$ since 4 is positive, but $|-2| = 2$ since -2 is negative.

We can also think of $|x|$ geometrically as the distance of x from 0 on the number line.



More generally, $|x - a|$ can be thought of as the distance of x from a on the numberline.



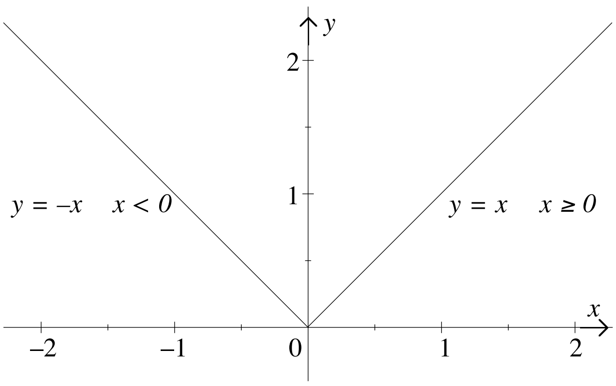
Note that $|a - x| = |x - a|$.

The absolute value *function* is written as $y = |x|$.

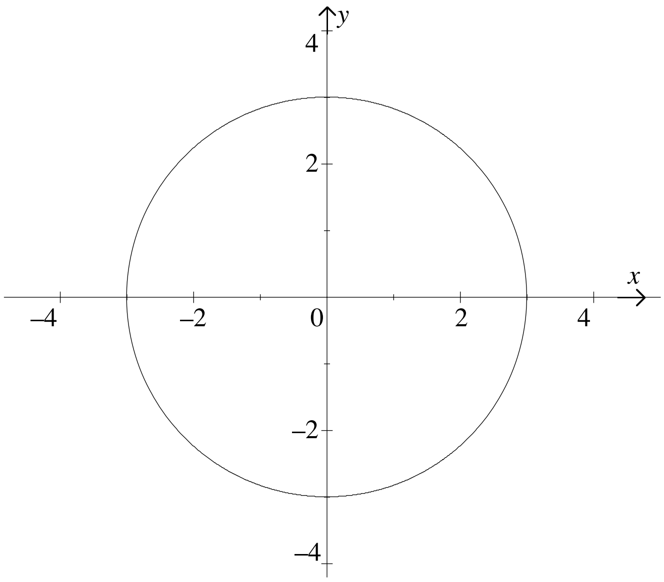
We define this function as

$$y = \begin{cases} +x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

From this definition we can graph the function by taking each part separately. The graph of $y = |x|$ is given below.

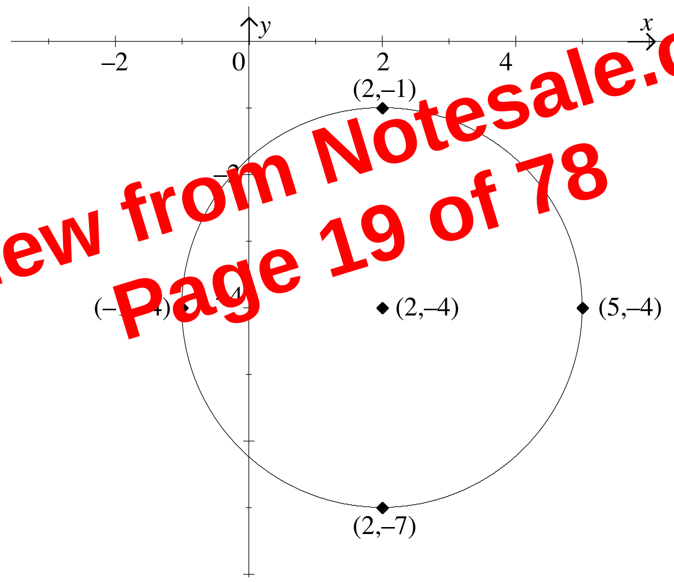


The graph of $y = |x|$.



The graph of $x^2 + y^2 = 9$.

This is a circle centre $(0, 0)$, radius 3.



The graph of $(x - 2)^2 + (y + 4)^2 = 9$.

This is a circle centre $(2, -4)$, radius 3.

Replacing x by $x - 2$ has the effect of shifting the graph of $x^2 + y^2 = 9$ two units to the right. Replacing y by $y + 4$ shifts it down 4 units.

2.6 Graphing by addition of ordinates

We can sketch the graph of functions such as $y = |x| + |x - 2|$ by drawing the graphs of both $y = |x|$ and $y = |x - 2|$ on the same axes then adding the corresponding y-values.

1. 1 solution if $k > 0$ or $k < -4$

2. 2 solutions if $k = 0$ or $k = -4$

3. 3 solutions if $-4 < k < 0$.

2.8 Exercises

1. Sketch the following:

a. $y = x^2$ **b.** $y = \frac{1}{3}x^2$ **c.** $y = -x^2$ **d.** $y = (x+1)^2$

2. Sketch the following:

a. $y = \frac{1}{x}$ **b.** $y = \frac{1}{x-2}$ **c.** $y = \frac{-2}{x}$ **d.** $y = \frac{1}{x+1} + 2$

3. Sketch the following:

a. $y = x^3$ **b.** $y = |x^3 - 2|$ **c.** $y = 3 - (x-1)^3$

4. Sketch the following:

a. $y = |x|$ **b.** $y = 2|x-2|$ **c.** $y = 4 - |x|$

5. Sketch the following:

a. $x^2 + y^2 = 16$ **b.** $x^2 + (y-2)^2 = 16$ **c.** $(x-1)^2 + (y-3)^2 = 16$

6. Sketch the following:

a. $y = \sqrt{9 - x^2}$ **b.** $y = \sqrt{9 - (x-1)^2}$ **c.** $y = \sqrt{9 - x^2} - 3$

7. Show that $\dfrac{x-1}{x-2} = \dfrac{1}{x-2} + 1$.

Hence sketch the graph of $y = \dfrac{x-1}{x-2}$.

8. Sketch $y = \frac{x+1}{x-1}$.

9. Graph the following relations in the given interval:

a. $y = |x| + x + 1$ for $-2 \le x \le 2$ [Hint: Sketch by adding ordinates]

b. $y = |x| + |x-1|$ for $-2 \le x \le 3$

c. $y = 2^x + 2^{-x}$ for $-2 \le x \le 2$

d. $|x - y| = 1$ for $-1 \le x \le 3$.

10. Sketch the function $f(x) = |x^2 - 1| - 1$.

Example

Decide whether the following functions are even, odd or neither.

1. $f(x) = 3x^2 - 4$

2. $g(x) = \frac{1}{2x}$

3. $f(x) = x^3 - x^2$.

Solution

1.

$$f(-x) = 3(-x)^2 - 4 = 3x^2 - 4 = f(x)$$

The function $f(x) = 3x^2 - 4$ is even.

2.

$$g(-x) = \frac{1}{2(-x)} = \frac{1}{-2x} = -\frac{1}{2x} = -g(x)$$

Therefore, the function g is odd.

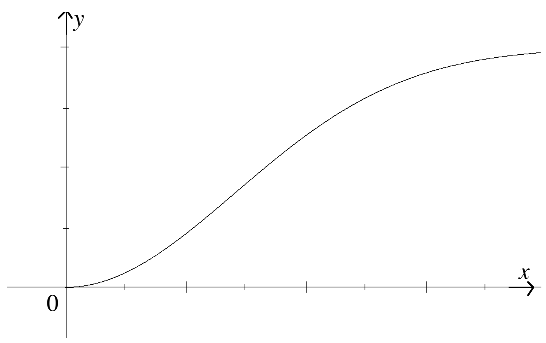
3.

$$f(-x) = (-x)^3 - (-x)^2 = -x^3 - x^2$$

This function is neither even (since $-x^3 - x^2 \neq x^3 - x^2$) nor odd (since $-x^3 - x^2 \neq -(x^3 - x^2)$).

Example

Sketched below is part of the graph of $y = f(x)$.



Complete the graph if $y = f(x)$ is

1. odd

2. even.

3 Piecewise functions and solving inequalities

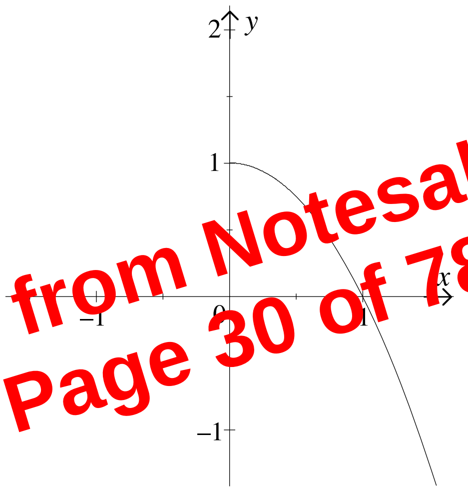
In this Chapter we will discuss functions that are defined piecewise (sometimes called piecemeal functions) and look at solving inequalities using both algebraic and graphical techniques.

3.1 Piecewise functions

3.1.1 Restricting the domain

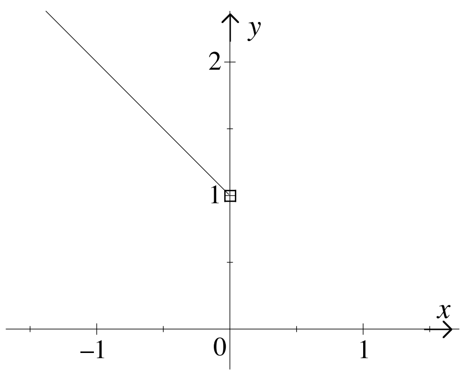
In Chapter 1 we saw how functions could be defined on a subinterval of their natural domain. This is frequently called *restricting* the domain of the function. In this Chapter we will extend this idea to define functions piecewise.

Sketch the graph of $y = 1 - x^2$ for $x \geq 0$.



The graph of $y = 1 - x^2$ for $x \geq 0$.

Sketch the graph of $y = 1 - x$ for $x < 0$.



The graph of $y = 1 - x$ for $x < 0$.

iii. $y = x + 3$ intersects $y = |2x - 6|$ twice.
To solve $|2x - 6| = x + 3$, take $|2x - 6| = 2x - 6 = x + 3$ when $x \geq 3$. This gives us the solution $x = 9$. Then take $|2x - 6| = -2x + 6 = x + 3$ when $x < 3$ which gives us the solution $x = 1$.

iv. When is the absolute value graph below the line $y = x + 3$?
From the graph, $1 < x < 9$.

v. $y = x - 3$ intersects the absolute value graph at $x = 3$ only.

b. k represents the y-intercept of the line $y = x + k$. When $k = -3$, there is one point of intersection. (See (a) (v) above). For $k > -3$, lines of the form $y = x + k$ will have two points of intersection. Hence $|2x - 6| = x + k$ will have two solutions for $k > -3$.

3.4 Exercises

1. Solve

a. $x^2 \leq 4x$

b. $\frac{4p}{p+3} \leq 1$

c. $\frac{7}{9-x^2} > -1$

2. **a.** Sketch the graph of $y = 4x(x - 3)$.

b. Hence solve $4x(x - 3) \leq 0$.

3. **a.** Find the points of intersection of the graphs $y = 5 - x$ and $y = \frac{4}{x}$.

b. On the same set of axes, sketch the graphs of $y = 5 - x$ and $y = \frac{4}{x}$.

c. Using part (ii), or otherwise, write down all the values of x for which

$$5 - x > \frac{4}{x}$$

4. **a.** Sketch the graph of $y = 2^x$.

b. Solve $2^x < \frac{1}{2}$.

c. Suppose $0 < a < b$ and consider the points A$(a, 2^a)$ and B$(b, 2^b)$ on the graph of $y = 2^x$. Find the coordinates of the midpoint M of the segment AB.
Explain why

$$\frac{2^a + 2^b}{2} > 2^{\frac{a+b}{2}}$$

5. **a.** Sketch the graphs of $y = x$ and $y = |x - 5|$ on the same diagram.

b. Solve $|x - 5| > x$.

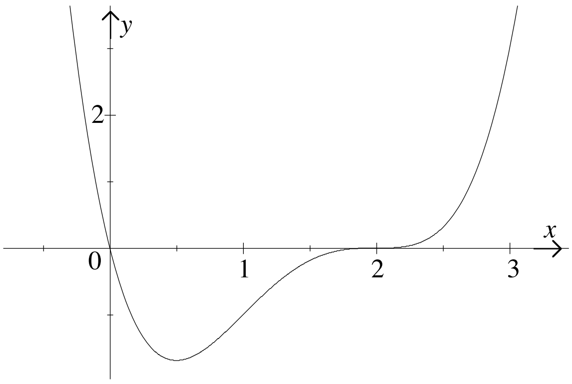
c. For what values of m does $mx = |x - 5|$ have exactly

i. two solutions

ii. no solutions

6. Solve $5x^2 - 6x - 3 \leq |8x|$.

We illustrate **3.** by sketching the graph of $f(x) = x(x-2)^3$. Notice the horizontal point of inflection at $x = 2$.



The graph of $f(x) = x(x-2)^3$.

4.3 Exercises

1. Sketch the graphs of the following polynomials if $y = P(x)$ is:

a. $x(x+1)(x-3)$

b. $x(x+1)(3-x)$

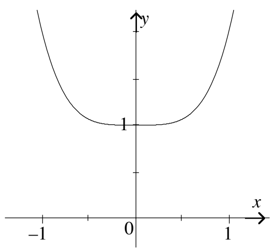
c. $(x+1)^2(x-3)$

d. $(x+1)(x^2-4x+5)$

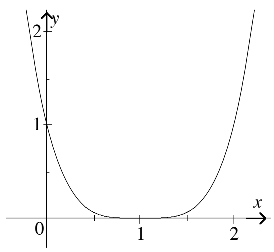
2. The graphs of the following quartic polynomials are sketched below. Match the graph with the polynomial.

a. $y = x^4$ **b.** $y = x^4 - 1$ **c.** $y = x^4 + 1$ **d.** $y = 1 - x^4$ **e.** $y = (x-1)^4$ **f.** $y = (x+1)^4$

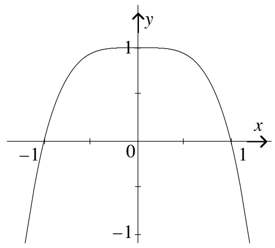
i.



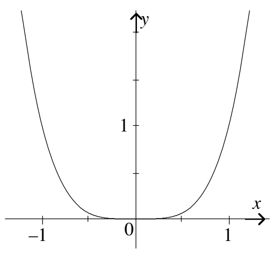
ii.



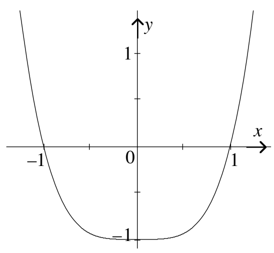
iii.



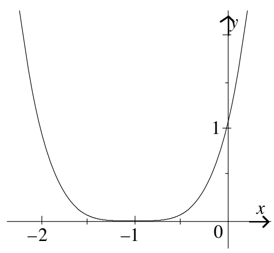
iv.



v.



vi.

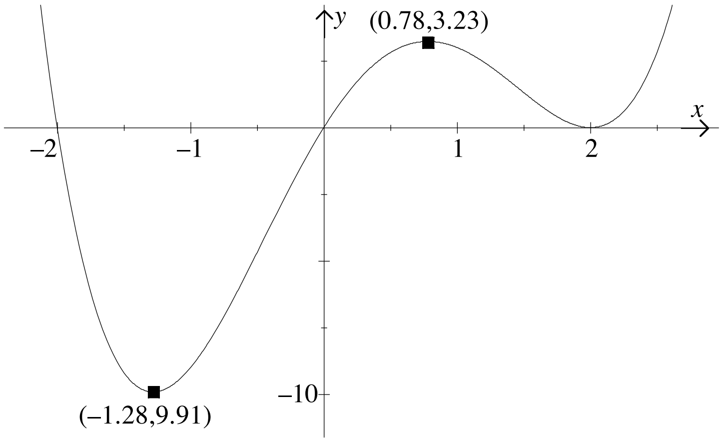


d. Solve the inequality $f(x) < 0$.

e. What is the *least* possible degree of $f(x)$?

f. State the value of the constant of $f(x)$.

g. For what values of k is $f(x) + k \geq 0$ for all real x.



The graph of the polynomial $y = f(x)$

4.4 Factorising polynomials

So far for the most part, we have looked at polynomials which were already factorised. In this section we will look at methods which will help us factorise polynomials with degree > 2.

4.4.1 Dividing polynomials

Suppose we have two polynomials $P(x)$ and $A(x)$, with the degree of $P(x) \geq$ the degree of $A(x)$, and $P(x)$ is divided by $A(x)$. Then

$$\frac{P(x)}{A(x)} = Q(x) + \frac{R(x)}{A(x)},$$

where $Q(x)$ is a polynomial called the *quotient* and $R(x)$ is a polynomial called the *remainder*, with the degree of $R(x) <$ degree of $A(x)$.

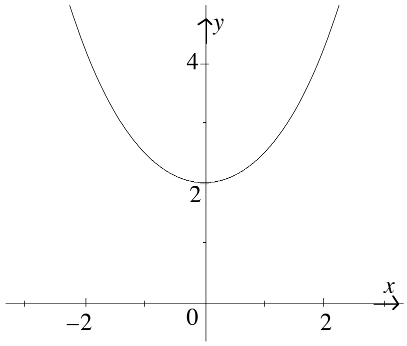
We can rewrite this as

$$P(x) = A(x) \cdot Q(x) + R(x).$$

For example: If $P(x) = 2x^3 + 4x + 3$ and $A(x) = x - 2$, then $P(x)$ can be divided by $A(x)$ as follows:

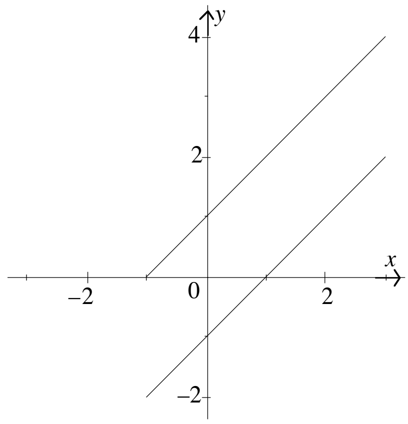
$$\begin{array}{r|rrrr}
 & 2x^2 + & 4x + & 12 & \\
\cline{2-5}
x - 2 & 2x^3 + 0x^2 + & 4x - & 3 & \\
 & 2x^3 - 4x^2 & & & \\
\cline{2-5}
 & 4x^2 + & 4x - & 3 & \\
 & 4x^2 - & 8x & & \\
\cline{3-5}
 & & 12x - & 3 & \\
 & & 12x - & 24 & \\
\cline{3-5}
 & & & 21 &
\end{array}$$

c.



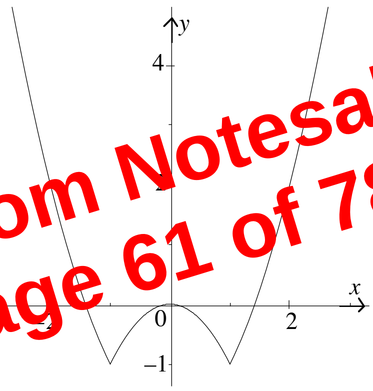
The graph of $y = 2^x + 2^{-x}$ for $-2 \le x \le 2$.

d.



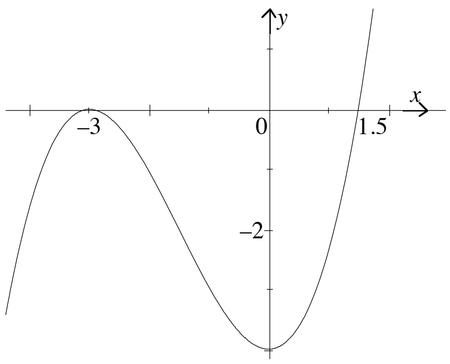
The graph of $|x - y| = 1$ for $-1 \le x \le 3$.

10.



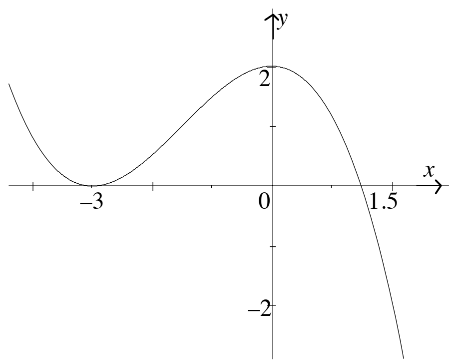
The graph of $f(x) = |x^2 - 1| - 1$.

11. a.



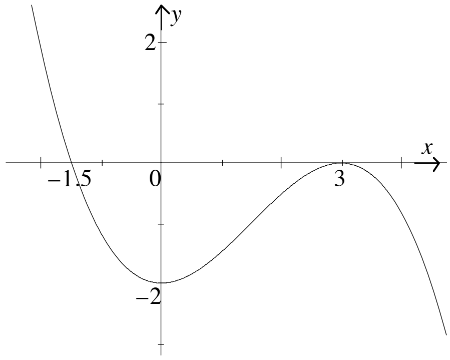
The graph of $y = 2f(x)$.

b.



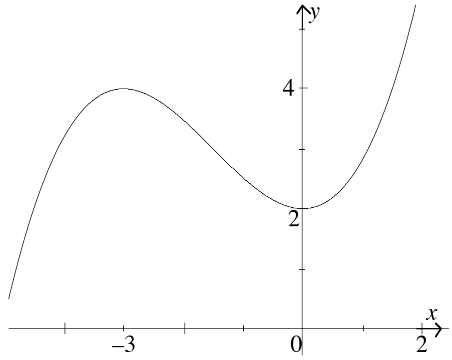
The graph of $y = -f(x)$.

c.



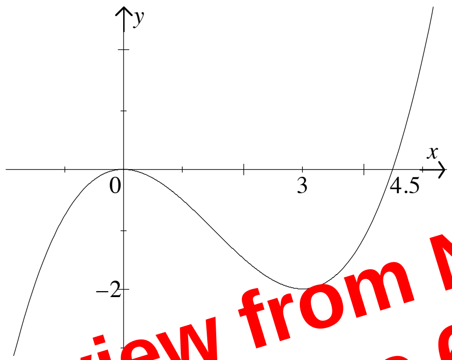
The graph of $y = f(-x)$.

d.



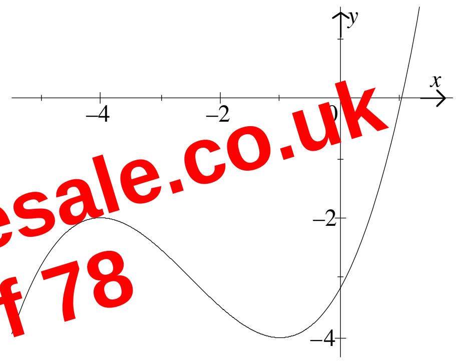
The graph of $y = f(x) + 4$.

e.



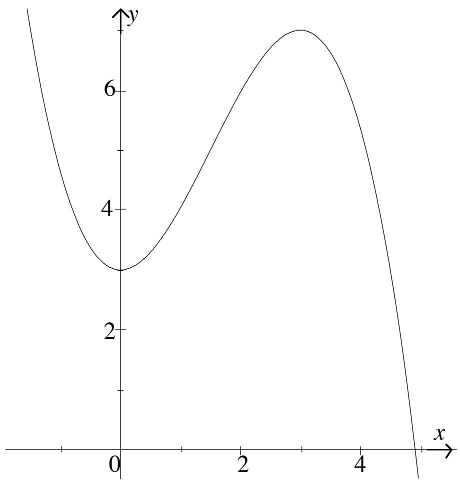
The graph of $y = f(x - 3)$.

f.



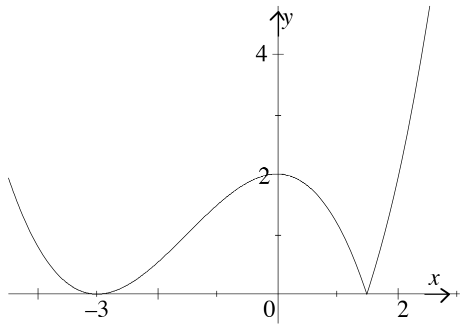
The graph of $y = f(x + 1) - 2$.

g.



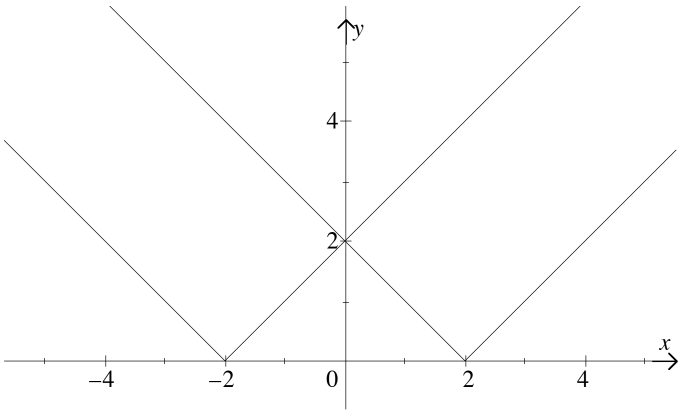
The graph of $y = 3 - 2f(x - 3)$.

h.



The graph of $y = |f(x)|$.

17.

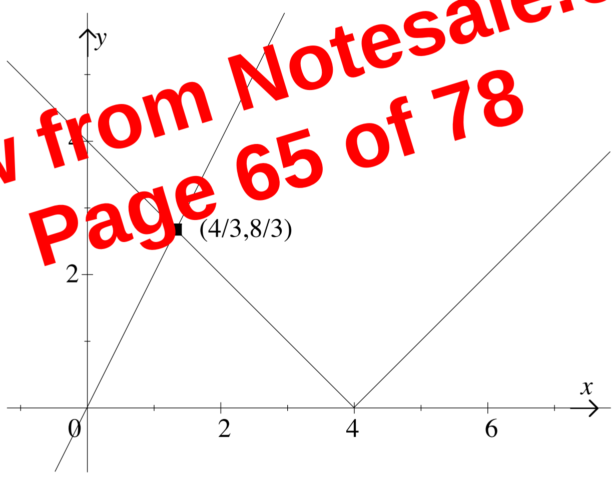


The point of intersection is $(0, 2)$. Therefore the solution of $|x-2| = |x+2|$ is $x = 0$.

18. $n = -1$ or $n = 2$.

19. a. For $x \geq 4$, $|x-4| = x-4 = 2x$ when $x = -4$, but this does not satisfy the condition of $x \geq 4$ so is not a solution.
For $x < 4$, $|x-4| = -x+4 = 2x$ when $x = \frac{4}{3}$. $x = \frac{4}{3}$ is < 4 so is a solution.
Therefore, $x = \frac{4}{3}$ is a solution of $|x-4| = 2x$.

b.



The graph of $y = |x-4|$ and $y = 2x$ intersect at the point $(\frac{4}{3}, \frac{8}{3})$. So the solution of $|x-4| = 2x$ is $x = \frac{4}{3}$.

2.11 Solutions

1. a. The domain is all real x, and the range is all real $y \geq -2$.

b. i. $-2 < x < 0$ or $x > 2$

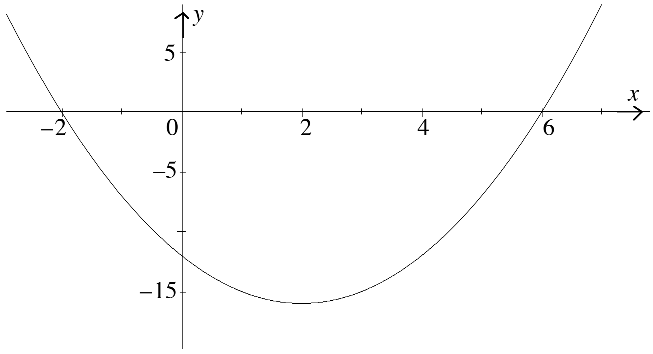
ii. $x < -2$ or $0 < x < 2$

c. i. $k < -2$

ii. There is no value of k for which $f(x) = k$ has exactly one solution.

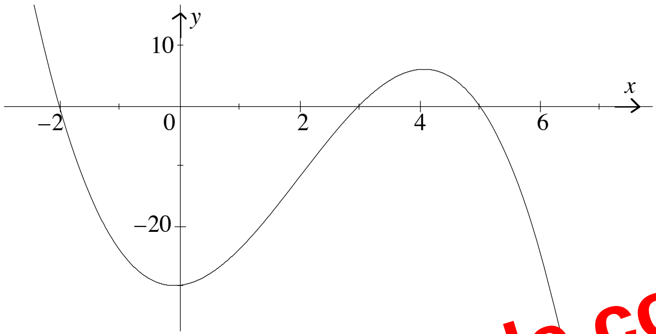
iii. $k = 2$ or $k > 0$

4. **a.**



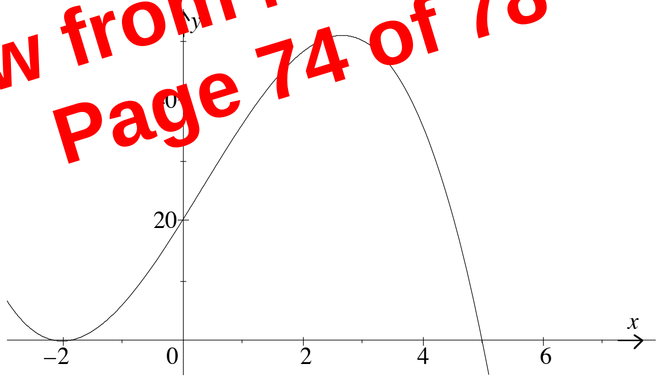
$x^2 - 4x - 12 < 0$ when $-2 < x < 6$.

b.



$(x+2)(x-3)(5-x) > 0$ when $x < -2$ or $3 < x < 5$.

c.



$(x+2)^2(5-x) > 0$ when $x < 5$.

5. $x^2 - 4x - 12 + k \geq 0$ for all real x when $k = 16$.

6. **a.** $P(x) = x(x-4)$

b. $P(x) = -x(x-4)$

c. $P(x) = x^2(x-4)$

d. $P(x) = \frac{x^3(x-4)}{3}$

e. $P(x) = -x(x-4)^2$

f. $P(x) = \frac{(x+4)(x-4)^2}{8}$

7. **a.** The roots of $f(x) = 0$ are $x = -2$, $x = 0$ and $x = 2$.