$$\beta_2 = 2\beta_2' = \rho_2(\sigma_2 - 1) + \frac{\rho_2}{2},$$
$$\gamma_2 = 2\gamma_2' = \rho_2\sigma_2.$$

Then Eqs. (36)–(39) agree with Eqs. (24)–(27) for $i = 1, 2$ and $j = 1, 2, \ldots, 2^n$. Next, subtracting equations corresponding to $p = k, q = 2l$ from equations corresponding to $p = k, q = 2l - 1$, respectively, for $k = 1, 2$ and $l = 1, 2, \ldots, 2^n$, we obtain the following system:

$$\begin{aligned} F_{1,2l-1} - F_{1,2l} &= 2b_{1,2^n+l} + 2b_{2,2^n+l}, \quad l = 1, 2, \ldots, 2^n \\ F_{2,2l-1} - F_{2,2l} &= 2b_{1,2^n+l} - 2b_{2,2^n+l}, \quad l = 1, 2, \ldots, 2^n. \end{aligned} \tag{40}$$

This implies the following:

$$\begin{aligned} b_{1,2^n+l} &= \frac{1}{4}\left(F_{1,2l-1} + F_{2,2l-1} - F_{1,2l} - F_{1,2l}\right), \quad l = 1, 2, \ldots, 2^n \\ b_{2,2^n+l} &= \frac{1}{4}\left(F_{1,2l-1} - F_{1,2l} - F_{2,2l-1} + F_{2,2l}\right), \quad l = 1, 2, \ldots, 2^n. \end{aligned} \tag{41}$$

The values given in Eq. (41) agree with the values obtained from Eqs. (26) and (27). This proves the theorem when $m = 0$ and $n > 0$. A similar reasoning proves the theorem for $m > 0$ and $n = 0$.

Finally, we assume that $m > 0$ and $n > 0$. Adding equations corresponding to $p = 2k - 1, q = l$ and $p = 2k, q = l$ for $k = 1, 2, \ldots, 2^n$ and $l = 1, 2, \ldots, 2^{n+1}$, we obtain the following system:

$$G_{p,q} = \sum_{i=1}^{2^m}\sum_{j=1}^{2^{n+1}} b_{i,j}\,\mathrm{haar}_i(x_p)\,\mathrm{haar}_j(t_q), \quad p = 1, 2, \ldots, 2^m,\ q = 1, 2, \ldots, 2^{n+1}, \tag{42}$$

where

$$G_{p,q} = \frac{1}{2}\left(F_{2p-1,q} + F_{2p,q}\right). \tag{43}$$

Applying the induction hypothesis to system (42), we obtain the following values of $b_{i,j}$:

$$b_{1,1} = \frac{1}{2^m \times 2^{n+1}} \sum_{p=1}^{2^m}\sum_{q=1}^{2^{n+1}} G_{p,q}, \tag{44}$$

$$b_{i,1} = \frac{1}{\rho_1' \times 2^{n+1}}\left(\sum_{p=\alpha_1'}^{\beta_1'}\sum_{q=1}^{2^{n+1}} G_{p,q} - \sum_{p=\beta_1'+1}^{\gamma_1'}\sum_{q=1}^{2^{n+1}} G_{p,q}\right), \quad i = 2, 3, \ldots, 2^m, \tag{45}$$

$$b_{1,j} = \frac{1}{2^m \times \rho_2}\left(\sum_{p=1}^{2^m}\sum_{q=\alpha_2}^{\beta_2} G_{p,q} - \sum_{p=1}^{2^m}\sum_{q=\beta_2+1}^{\gamma_2} G_{p,q}\right), \quad j = 2, 3, \ldots, 2^n, \tag{46}$$

$$\begin{aligned} b_{i,j} = \frac{1}{\rho_1'\rho_2}&\left(\sum_{p=\alpha_1'}^{\beta_1'}\sum_{q=\alpha_2}^{\beta_2} G_{p,q} - \sum_{p=\alpha_1'}^{\beta_1'}\sum_{q=\beta_2+1}^{\gamma_2} G_{p,q} - \sum_{p=\beta_1'+1}^{\gamma_1'}\sum_{q=\alpha_2}^{\beta_2} G_{p,q}\right. \\ &\left. + \sum_{p=\beta_1'+1}^{\gamma_1'}\sum_{q=\beta_2+1}^{\gamma_2} G_{p,q}\right), \quad i = 2, 3, \ldots, 2^m,\ j = 2, 3, \ldots, 2^n, \end{aligned} \tag{47}$$

where $\tau_1' = 2^{\lfloor \log_2(i-1)\rfloor}$, $\sigma_1' = i - \tau_1'$, $\rho_1' = \frac{2^m}{\tau_1'}$, $\alpha_1' = \rho_1'(\sigma_1' - 1) + 1$, $\beta_1' = \rho_1'(\sigma_1' - 1) + \frac{\rho_1'}{2}$, $\gamma_1' = \rho_1'\sigma_1'$, $\tau_2' = 2^{\lfloor \log_2(j-1)\rfloor}$, $\sigma_2' = j - \tau_2'$, $\rho_2' = \frac{2^{n+1}}{\tau_2'}$, $\alpha_2' = \rho_2'(\sigma_2' - 1) + 1$, $\beta_2' = \rho_2'(\sigma_2' - 1) + \frac{\rho_2'}{2}$ and $\gamma_2' = \rho_2'\sigma_2'$. After simplifications, Eqs. (44)–(47) can be written as follows:

$$b_{1,1} = \frac{1}{2^{m+1} \times 2^{n+1}} \sum_{p=1}^{2^{m+1}}\sum_{q=1}^{2^{n+1}} F_{p,q}. \tag{48}$$

$$b_{i,1} = \frac{1}{2\rho_1' \times 2^{n+1}}\left(\sum_{p=2\alpha_1'-1}^{2\beta_1'}\sum_{q=1}^{2^{n+1}} F_{p,q} - \sum_{p=2\beta_1'+1}^{2\gamma_1'}\sum_{q=1}^{2^{n+1}} F_{p,q}\right), \quad i = 2, 3, \ldots, 2^m, \tag{49}$$

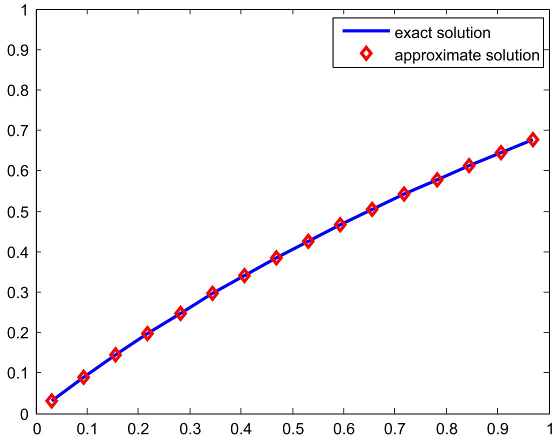

Fig. 4. Comparison of exact and approximate solutions for Example 4.

Example 4. Consider the following nonlinear Volterra integral equation [49]:

$$u(x) = f(x) + \int_0^x xt^2 (u(t))^2 \mathrm{d}t, \tag{68}$$

where

$$f(x) = \left(1 + \frac{11}{9}x + \frac{2}{3}x^2 - \frac{1}{3}x^3 + \frac{2}{9}x^4\right)\ln(x+1) - \frac{1}{3}(x + x^4)(\ln(x+1))^2 - \frac{11}{9}x^2 + \frac{5}{18}x^3 - \frac{2}{27}x^4. \tag{69}$$

The exact solution of this problem is $u(x) = \ln(x + 1)$. In Fig. 4, we have shown the comparison of the approximate solution with the exact solution.

5. Conclusion

Two new generic algorithm are proposed for the numerical solution of nonlinear Fredholm integral equations of the second kind and nonlinear Volterra integral equations of the second kind. A two-dimensional Haar wavelet basis is used for this purpose. The algorithms are established theoretically alongside numerical validations. The new algorithms do not need any linear system solution for evaluation of the wavelet coefficients and are more efficient than conventional Haar wavelet based methods. Different types of integral equation can be solved numerically by the same method more accurately than previously.

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