We have already noted that in order for (K, L, N)-TFFs to exist, one needs $KL \geqslant N$; we now use Corollary 8 to prove a stronger necessary condition on existence, given in Theorem 1:

Corollary 9. *If (K, L, N)-TFFs exist and L does not divide N, then $K \geqslant \lceil \frac{N}{L} \rceil + 1$.*

Proof. If (K, L, N)-TFFs exist, then $KL \geqslant N$. Since L does not divide N, then $KL > N$, and so $(K, L, KL - N)$-TFFs exist by the previous result. Thus, there exist L orthonormal vectors in $\mathbb{C}^{KL-N}$, and as such, $L \leqslant KL - N$. Simplifying, we find $K \geqslant \frac{N}{L} + 1$. Since K is an integer, taking the ceiling of both sides of this equation yields the result. □

We note that the necessary condition of Corollary 9 is not sufficient. In particular, $(3, 3, 4)$-TFFs do not exist, despite the fact that $3 \geqslant \lceil \frac{4}{3} \rceil + 1$. Indeed, if a $(3, 3, 4)$-TFF did exist, then its spatial complement, obtained by applying Corollary 8(i), would be a $(3, 1, 4)$-TFF; such TFFs do not exist by Corollary 9, since $3 < \lceil \frac{4}{1} \rceil + 1$.

One may preclude such simple counterexamples to the sufficiency of Corollary 9's condition by making the further requirement that $2L < N$. However, even in this case, $K \geqslant \lceil \frac{N}{L} \rceil + 1$ is not sufficient: $(4, 4, 11)$-TFFs do not exist, despite the fact that $4 \geqslant \lceil \frac{11}{4} \rceil + 1$ and $2(4) < 11$. To be precise, if a $(4, 4, 11)$-TFF did exist, then its Naimark complement, obtained by applying Corollary 8(ii), would be a $(4, 4, 5)$-TFF, whose spatial complement would, in turn, be a $(4, 1, 5)$-TFF; such frames do not exist since $4 < \lceil \frac{5}{1} \rceil + 1$.

To summarize, the conditions $2L < N$ and $K \geqslant \lceil \frac{N}{L} \rceil + 1$ are not sufficient to guarantee the existence of (K, L, N)-TFFs. However, one of the main results of this paper, as encapsulated in the final statement of Theorem 1, is to show that a very slight strengthening of these conditions is actually sufficient for existence. Specifically, over the course of the next two sections, we will provide an explicit construction of a (K, L, N)-TFF for each $K, L, N \in \mathbb{N}$ such that L does not divide N, $2L < N$ and $K \geqslant \lceil \frac{N}{L} \rceil + 2$. That is, we will show that TFFs indeed exist whenever the number of subspaces K is at least two more than what is absolutely necessary. Moreover, in the final section, we will show that the existence of equal-rank TFFs is completely resolved using this construction along with a finite number of repeated applications of Corollary 8.

3. Spectral Tetris

In this section, we provide the first half of a general method for constructing (K, L, N)-TFFs when $K \geqslant \lceil \frac{N}{L} \rceil + 2$. The key idea is to revisit the simpler problem of constructing UNTFs, that is, sequences $\{f_m\}_{m=1}^{M}$ of unit vectors in $\mathbb{C}^N$ that satisfy (2). In brief, we want to construct $N \times M$ synthesis matrices F which have:

(i) columns of unit norm,
(ii) orthogonal rows, meaning the frame operator FF^* is diagonal,
(iii) rows of constant norm, meaning FF^* is a constant multiple of the identity matrix.

Despite a decade of study, very few general constructions of finite-dimensional UNTFs are known. Moreover, these known methods unfortunately manipulate all frame elements simultaneously. In this section, we show that constructing certain examples of UNTFs need not be so difficult. In particular, we provide a new, iterative method for constructing UNTFs, building them one or two vectors at a time. The key idea is to iteratively build a matrix F which, at each iteration, exactly satisfies (i) and (ii), and gets closer to satisfying (iii). We call this method *Spectral Tetris*, as it involves building a flat spectrum out of blocks of fixed area. Here, an illustrative example is helpful:

Example 10. In the previous section, we showed that $(4, 4, 11)$-TFFs did not exist, despite the fact that these K, L and N satisfy the necessary condition for existence given in Corollary 9. At the same time, we claim in Theorem 1 that a slightly stronger requirement, $K \geqslant \lceil \frac{N}{L} \rceil + 2$, is indeed sufficient for existence, provided L does not divide N and $2L < N$. In particular, Theorem 1 asserts that $(5, 4, 11)$-TFFs exist. In this and the following sections, we will show how to explicitly construct such a TFF, so as to illustrate the simple ideas behind the proof of Theorem 1(ii). The construction is performed over two stages. The first stage, given in the present example, is to play Spectral Tetris, yielding a sparse UNTF of 11 elements for $\mathbb{C}^4$. In the second stage, this UNTF is then modulated to produce a $(5, 4, 11)$-TFF, as described in Example 15.

Our immediate goal is to create a 4×11 matrix F such that $FF^* = \frac{11}{4}\mathrm{I}$. As such, we begin with an arbitrary 4×11 matrix, and let the first two frame elements be copies of the first standard basis element e_1:

$$F = \begin{bmatrix} 1 & 1 & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? & ? & ? & ? & ? & ? \end{bmatrix}. \tag{6}$$

If the remaining unknown entries are chosen so that F has orthogonal rows, then FF^* will be a diagonal matrix. Currently, the diagonal entries of FF^* are mostly unknown, having the form $\{2+?, ?, ?, ?\}$. Also note that if the remainder of the first row of F is set to zero, then the first diagonal entry of FF^* would be $2 < \frac{11}{4}$. Thus, we need to add more weight to this row. However, making the third column of F another copy of e_1 would add too much weight, as $3 > \frac{11}{4}$. Therefore, we need

Table 1
The analysis operator of a $(5,4,11)$-TFF, as described in Example 15. Here, $w := \mathrm{e}^{-2\pi\mathrm{i}/5}$. The rows of this matrix form a TFF for $\mathbb{C}^{11}$ consisting of 5 subspaces, each of dimension 4. Here, a given pair of rows belong to the same subspace if their indices differ by a multiple of 5.

$$
\begin{bmatrix}
1 & 1 & \frac{\sqrt{3}}{\sqrt{8}} & \frac{\sqrt{3}}{\sqrt{8}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & w & \frac{\sqrt{3}}{\sqrt{8}}w^2 & \frac{\sqrt{3}}{\sqrt{8}}w^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & w^2 & \frac{\sqrt{3}}{\sqrt{8}}w^4 & \frac{\sqrt{3}}{\sqrt{8}}w^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & w^3 & \frac{\sqrt{3}}{\sqrt{8}}w & \frac{\sqrt{3}}{\sqrt{8}}w^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & w^4 & \frac{\sqrt{3}}{\sqrt{8}}w^3 & \frac{\sqrt{3}}{\sqrt{8}}w & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\sqrt{5}}{\sqrt{8}} & -\frac{\sqrt{5}}{\sqrt{8}} & 1 & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\sqrt{5}}{\sqrt{8}}w^2 & -\frac{\sqrt{5}}{\sqrt{8}}w^3 & w^4 & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}}w & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\sqrt{5}}{\sqrt{8}}w^4 & -\frac{\sqrt{5}}{\sqrt{8}}w & w^3 & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}}w^2 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\sqrt{5}}{\sqrt{8}}w & -\frac{\sqrt{5}}{\sqrt{8}}w^4 & w^2 & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}}w^3 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\sqrt{5}}{\sqrt{8}}w^3 & -\frac{\sqrt{5}}{\sqrt{8}}w^2 & w & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}}w^4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}} & 1 & \frac{\sqrt{7}}{\sqrt{8}} & \frac{\sqrt{7}}{\sqrt{8}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}}w & w^2 & \frac{\sqrt{7}}{\sqrt{8}}w^3 & \frac{\sqrt{7}}{\sqrt{8}}w^4 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}}w^2 & w^4 & \frac{\sqrt{7}}{\sqrt{8}}w & \frac{\sqrt{7}}{\sqrt{8}}w^3 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}}w^3 & w & \frac{\sqrt{7}}{\sqrt{8}}w^4 & \frac{\sqrt{7}}{\sqrt{8}}w^2 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}}w^4 & w^3 & \frac{\sqrt{7}}{\sqrt{8}}w^2 & \frac{\sqrt{7}}{\sqrt{8}}w & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{8}} & -\frac{\sqrt{7}}{\sqrt{8}} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{8}}w^3 & -\frac{\sqrt{7}}{\sqrt{8}}w^4 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{8}}w & -\frac{\sqrt{7}}{\sqrt{8}}w^3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{8}}w^4 & -\frac{\sqrt{7}}{\sqrt{8}}w^2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{8}}w^2 & -\frac{\sqrt{7}}{\sqrt{8}}w & 1
\end{bmatrix}
$$

Table 2
The Tight Fusion Frame Existence Test (TFFET). As shown in the proof of Theorem 2, applying this test to any given $K, L, N \in \mathbb{N}$, $L < N$, will resolve the existence of (K, L, N)-TFFs in no more than L iterations of its “while” loop.

```
01 set K,L,N ∈ ℕ, L < N
02 if 2L > N, L := N − L
03 exists := 'unknown'
04 while exists := 'unknown'
05   if L | N
06     if K ⩾ N/L, exists := 'true'
07     else exists := 'false'
08   else
09     if K > ⌈N/L⌉+1, exists := 'true'
10     else if K < ⌈N/L⌉+1, exists := 'false'
11     else N := KL − N, L := N − L
12 end while
```

In this case, we necessarily have $L_j < KL_j - N_j$, and so we can apply Corollary 8(ii) and then Corollary 8(i) to obtain that (K, L_j, N_j)-TFFs exist if and only if $(K, L_{j+1}, N_{j+1}) := (K, (K-1)L_j - N_j, KL_j - N_j)$-TFFs exist. In TFFET, the reduction of (K, L_j, N_j) to (K, L_{j+1}, N_{j+1}) is accomplished in Line 11. In essence, TFFET's “while” loop first checks whether Theorem 1 resolves the existence of (K, L_j, N_j)-TFFs; in the case where it does not, TFFET instead calculates the alternative triple (K, L_{j+1}, N_{j+1}) for which the question of TFF existence is equivalent to that of the original. Note that the full utility of Theorem 1 is predicated upon whether $2L < N$; it is therefore important to note that whenever a given triple (K, L_j, N_j) is ambiguous, we have $K = \lceil \frac{N_j}{L_j} \rceil + 1 < \frac{N_j}{L_j} + 2$, and so $2L < N$ also holds for the new triple:

$$
\begin{aligned}
2L_{j+1} &= 2\big[(K-1)L_j - N_j\big] = KL_j + \big[(K-2)L_j - 2N_j)\big] < \left(\frac{N_j}{L_j} + 2\right)L_j + \big[(K-2)L_j - 2N_j)\big] \\
&= KL_j - N_j = N_{j+1}.
\end{aligned}
$$