Cambridge International Examination
General Certificate of Education Advanced Subsidiary Level
Mathematics
Paper 2 Pure mathematics (P2)
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Q7(i) Show that $\int_0^{\frac{\pi}{4}} \sin 2x \, dx = \frac{1}{2}$ and that $\int_0^{\frac{\pi}{4}} \cos^2 x \, dx = \frac{1}{8}(\pi + 2)$

(ii) Use the result in part (i) to evaluate $\int_0^{\frac{\pi}{4}} (2\sin x + 3\cos x)^2 dx$

Solution (i)

Since $\sin 2x = 2\sin x\cos x$

Also $\frac{d}{dx}\sin x = \cos x$

$d(\sin x) = \cos x \, dx$

$$\text{Now} \int_0^{\frac{\pi}{4}} \sin 2x dx = \int_0^{\frac{\pi}{4}} 2\sin x\cos x \, dx = 2\int_0^{\frac{\pi}{4}} \sin x \, (\cos x dx)$$

$$= 2\int_0^{\frac{\pi}{4}} \sin x \; d(\sin x) = 2\left\{\frac{(\sin x)^2}{2}\right\}_0^{\frac{\pi}{4}} = \left[(\sin x)^2\right]_0^{\frac{\pi}{4}} = \left(\sin\frac{\pi}{4}\right)^2 - (\sin 0)^2$$

$$= \left(\frac{\sqrt{2}}{2}\right)^2 - 0 = \frac{1}{2}$$

$$\text{Hence} \int_0^{\frac{\pi}{4}} \sin 2x \, dx = \frac{1}{2}$$

Also $\cos 2x = \cos^2 x - \sin^2 x$ ------------ (1)

$\sin^2 x + \cos^2 x = 1$

$\Longrightarrow \sin^2 x = 1 - \cos^2 x$ ------------ (2)

From equation (1) and equation (2)

$\cos 2x = \cos^2 x - (1 - \cos^2 x) = \cos^2 x - 1 + \cos^2 x = 2\cos^2 x - 1$

$\cos 2x + 1 = 2\cos^2 x$

$\frac{1 + \cos 2x}{2} = \cos^2 x$

$$\Longrightarrow \int_0^{\frac{\pi}{4}} \cos^2 x dx = \int_0^{\frac{\pi}{4}} \frac{1 + \cos 2x}{2} dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} dx + \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos 2x \, dx$$

$$= \frac{1}{2}[x]_0^{\frac{\pi}{4}} + \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\frac{\pi}{4} + \frac{1}{4}\left[\sin 2\left(\frac{\pi}{4}\right) - \sin 2(0)\right]$$

$$= \frac{\pi}{8} + \frac{1}{4}\left[\sin\frac{\pi}{2} - \sin(0)\right] = \frac{\pi}{8} + \frac{1}{4}(1) = \frac{\pi}{8} + \frac{2}{8}$$

$$\Longrightarrow \int_0^{\frac{\pi}{4}} \cos^2 x \, dx = \frac{1}{8}(\pi + 2)$$

(ii)

$$I = \int_0^{\frac{\pi}{4}} (2\sin x + 3\cos x)^2 \, dx \quad \text{------------ (1)}$$

Now $(2\sin x + 3\cos x)^2 = 4\sin^2 x + 9\cos^2 x + 12\sin x\cos x$

$= 4\sin^2 x + 9\cos^2 x + 6\,(2\sin x\cos x)$ --------- (2)

Also

$\sin^2 x + \cos^2 x = 1$

$\Longrightarrow \sin^2 x = 1 - \cos^2 x$ ----------- (3)

$2\sin x\cos x = \sin 2x$ --------------- (4)

Putting equation (3) & equation (4) in equation (2)

$(2\sin x + 3\cos x)^2 = 4(1 - \cos^2 x) + 9\cos^2 x + 6\sin 2x$

$= 4 - 4\cos^2 x + 9\cos^2 x + 6\sin 2x$

$= 4 + 5\cos^2 x + 6\sin 2x$ ---------- (5)

Putting equation (5) in equation (1)

$$I = \int_0^{\frac{\pi}{4}} \left(4 + 5\cos^2 x + 6\sin 2x\right) dx$$