

Q7(i) Show that $\int_0^{\frac{\pi}{4}} \sin 2x \, dx = \frac{1}{2}$ and that $\int_0^{\frac{\pi}{4}} \cos^2 x \, dx = \frac{1}{8}(\pi + 2)$

(ii) Use the result in part (i) to evaluate $\int_0^{\frac{\pi}{4}} (2\sin x + 3\cos x)^2 \, dx$

Solution (i)

Since $\sin 2x = 2\sin x \cos x$

Also $\frac{d}{dx} \sin x = \cos x$

$d(\sin x) = \cos x \, dx$

Now $\int_0^{\frac{\pi}{4}} \sin 2x \, dx = \int_0^{\frac{\pi}{4}} 2\sin x \cos x \, dx = 2 \int_0^{\frac{\pi}{4}} \sin x (\cos x \, dx)$

$$= 2 \int_0^{\frac{\pi}{4}} \sin x \, d(\sin x) = 2 \left[\frac{(\sin x)^2}{2} \right]_0^{\frac{\pi}{4}} = [(\sin x)^2]_0^{\frac{\pi}{4}} = \left(\sin \frac{\pi}{4} \right)^2 - (\sin 0)^2$$

$$= \left(\frac{\sqrt{2}}{2} \right)^2 - 0 = \frac{1}{2}$$

Hence $\int_0^{\frac{\pi}{4}} \sin 2x \, dx = \frac{1}{2}$

Also $\cos 2x = \cos^2 x - \sin^2 x$ ----- (1)

$\sin^2 x + \cos^2 x = 1$ ----- (2)

From equation (1) and equation (2)

$\cos 2x + \cos^2 x - (1 - \cos^2 x) = \cos^2 x - 1 + \cos^2 x = 2\cos^2 x - 1$

$\cos 2x + 1 = 2\cos^2 x$

$\frac{1 + \cos 2x}{2} = \cos^2 x$

$\Rightarrow \int_0^{\frac{\pi}{4}} \cos^2 x \, dx = \int_0^{\frac{\pi}{4}} \frac{1 + \cos 2x}{2} \, dx = \frac{1}{2} \int_0^{\frac{\pi}{4}} 1 \, dx + \frac{1}{2} \int_0^{\frac{\pi}{4}} \cos 2x \, dx$

$$= \frac{1}{2} [x]_0^{\frac{\pi}{4}} + \frac{1}{2} \left[\frac{\sin 2x}{2} \right]_0^{\frac{\pi}{4}} = \frac{1}{2} \cdot \frac{\pi}{4} + \frac{1}{4} \left[\sin 2 \left(\frac{\pi}{4} \right) - \sin 2(0) \right]$$

$$= \frac{\pi}{8} + \frac{1}{4} \left[\sin \frac{\pi}{2} - \sin(0) \right] = \frac{\pi}{8} + \frac{1}{4} (1) = \frac{\pi}{8} + \frac{2}{8}$$

$\Rightarrow \int_0^{\frac{\pi}{4}} \cos^2 x \, dx = \frac{1}{8}(\pi + 2)$

(ii)

$I = \int_0^{\frac{\pi}{4}} (2\sin x + 3\cos x)^2 \, dx$ ----- (1)

Now $(2\sin x + 3\cos x)^2 = 4\sin^2 x + 9\cos^2 x + 12\sin x \cos x$
 $= 4\sin^2 x + 9\cos^2 x + 6(2\sin x \cos x)$ ----- (2)

Also

$\sin^2 x + \cos^2 x = 1$

$\Rightarrow \sin^2 x = 1 - \cos^2 x$ ----- (3)

$2\sin x \cos x = \sin 2x$ ----- (4)

Putting equation (3) & equation (4) in equation (2)

$(2\sin x + 3\cos x)^2 = 4(1 - \cos^2 x) + 9\cos^2 x + 6\sin 2x$

$= 4 - 4\cos^2 x + 9\cos^2 x + 6\sin 2x$

$= 4 + 5\cos^2 x + 6\sin 2x$ ----- (5)

Putting equation (5) in equation (1)

$I = \int_0^{\frac{\pi}{4}} (4 + 5\cos^2 x + 6\sin 2x) \, dx$