1.4 Union, Intersections, Complements

The *union* of sets x and y is the set obtained by pooling together their elements into one set. The union is denoted

$$x \cup y.$$

For example,

$$\{1, 3, 5\} \cup \{3, 5, 6, 7\} = \{1, 3, 5, 6, 7\}.$$

One can do unions of any family of sets. For example,

$$\{1\} \cup \{1, 2\} \cup \{1, 2, 3\} \cup \ldots = \{1, 2, 3, \ldots\}.$$

The *intersection* of sets x and y is the set containing the elements that are both in x and in y, and is denoted

$$x \cap y.$$

For example,

$$\{1, 3, 5, 6\} \cap \{2, 4, 6, 3, 8\} = \{3, 6\}.$$

The intersection can be empty of course:

$$\{2, 4, 5\} \cap \{1, 3, 7\} = \emptyset.$$

If the intersection of sets x and y is empty we say that x and y are *disjoint*.

One can take intersections of more than two sets as well:

$$\{1, 5, 3, 6\} \cap \{2, 3, 4\} \cap \{3, 8, 9\} = \{3\}.$$

Sometimes we are working within one fixed big set X. Then the *complement* of any given subset $A \subset X$ is the set of elements of X not in A:

$$A^c = \{p \in X \;:\; p \notin A\}.$$

1.5 Integers and Rationals

The numbers $0, 1, 2, 3, \ldots$ along with their negatives form the set $\mathbb{Z}$ of *integers*:

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}. \tag{1.5}$$

This is called the *Cartesian product* of A with B. For example,

$$\{2, 5, 6\} \times \{d, g\} = \{(2, d), (2, g), (5, d), (5, g), (6, d), (6, g)\}.$$

The Cartesian product of a set A with itself is denoted A^2:

$$A^2 = A \times A. \tag{1.9}$$

Thus the *plane*, coordinatized by real numbers, can be modeled mathematically as

$$\mathbb{R}^2 = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}. \tag{1.10}$$

1.7 Mappings and Functions

In calculus we work with *functions* specified by formulas such as

$$y = x^3 + x^2 + 1.$$

This relation is not read as simply an equality of two quantities y and $x^3 + x^2 + 1$, but rather as a procedure for computing one quantity from the value of another:

given the value $x = 2$ we compute $y = 2^3 + 2^2 + 1 = 13$.

Thus what we have here is a prescription: an input value for x leads to an ouput value y. Of course, the letters x and y are in themselves of no significance; the same function is specified by

$$s = t^4 + t^2 + 1.$$

Sometimes a function is specified not by a formula but by an explicit description; for example,

$$1_{\text{prime}}(m) = \begin{cases} 1 & \text{if } m \text{ is a prime number;} \\ 0 & \text{if } m \text{ is not a prime number.} \end{cases}$$

specifies a 'function of m', where m runs over the positive integers. For example,

$$1_{\text{prime}}(5) = 1, \quad \text{and} \quad 1_{\text{prime}}(4) = 0.$$

Then for any point P on the line l we can think of the geometical concept of the ratio

$$OP/OU,$$

where we take this to be negative if P is on the opposite side of O from U. Such ratios can be added and multiplied by using geometric constructions (these geometric operations on segments were described in Euclid's *Elements*). Thus they form a system of numbers called the *real numbers*.

For example, if P is just the point U then the ratio

$$OP/OU = OU/OU$$

corresponds to the number 1. Similarly, we have points P for which OP/OU is a rational such as $-4/7$. But there are also points P for which OP/OU cannot be expressed as a ratio of integers.

For example, consider a right angled triangle that has two sides of length OU. Then the diagonal is, by Pythagoras' theorem, has the ratio to OU given by $\sqrt{2}$. It is a fact that $\sqrt{2}$ *is not a rational number*, in that there is no rational number whose square is 2.

The rationals are *dense* in the real line: between any two distinct reals there lies a rational. The irrationals are also dense in the real line: between any two distinct reals lies an irrational.

2.2 The Extended Real Line

The extended real line is obtained by a largest element ∞, and a smallest element $-\infty$, to the real line $\mathbb{R}$:

$$\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\} \tag{2.1}$$

Here ∞ and $-\infty$ are abstract elements. We extend the order relation to $\mathbb{R}$ by declaring that

$$-\infty < x < \infty \qquad \text{for all } x \in \mathbb{R} \tag{2.2}$$

Much of our work will be on $\mathbb{R}^*$, instead of just $\mathbb{R}$.

We define addition on $\mathbb{R}^*$ as follows:

$$\begin{aligned} x + \infty &= \infty = \infty + x \quad \text{for all } x \in \mathbb{R}^* \text{ with } x > -\infty & (2.3)\\ y + (-\infty) &= -\infty = (-\infty) + y \quad \text{for all } y \in \mathbb{R}^* \text{ with } y < \infty. & (2.4) \end{aligned}$$

Thus,

$$\inf S \leq p \leq \sup S \quad \text{for all } p \in S.$$

If S contains just one point then the inf and sup coincide: for example,

$$\inf\{3\} = 3 = \sup\{3\}.$$

On the other hand

$$\inf S < \sup S \quad \text{if } S \text{ contains more than one point.} \tag{3.3}$$

Now consider another situation. Consider sets

$$B \subset A \subset \mathbb{R}^*.$$

Thus everything in B is also in A. Any upper bound of A is $\geq$ all elements of A and hence is $\geq$ all elements of B. Thus:

$$\text{every upper bound of } A \text{ is an upper bound of } B.$$

In particular,

$$\text{the least upper bound of } A \text{ is an upper bound of } B.$$

In other words,

$$\sup A \text{ is an upper bound of } B.$$

So, of course,

$$\text{the least upper bound of } B \text{ is } \leq \sup A.$$

Thus,

$$\sup B \leq \sup A \qquad \text{if } B \subset A. \tag{3.4}$$

Picking a smaller set decreases the supremum, where smaller means that it is contained in the larger set. ('Decreases' is in a lose sense here, as it may happen that $\sup A$ is equal to $\sup B$.)

By a similar reasoning we have

$$\inf A \leq \inf B \qquad \text{if } B \subset A. \tag{3.5}$$

Picking a smaller set increases the infimum, with qualifiers as before.

Chapter 4

Neighborhoods, Open Sets and Closed Sets

In this chapter we study some useful concepts for studying the concept of nearness of points in $\mathbb{R}^*$.

4.1 Intervals

An *interval* in $\mathbb{R}^*$ is, geometrically, just a segment in the extended real line. For example, all the points $x \in \mathbb{R}^*$ for which $1 \leq x \leq 2$ is an interval. More officially, an interval J is a non-empty subset of $\mathbb{R}^*$ with the property that for any two points of J all points between the two points also lie in J: if $s, t \in J$, with $s < t$, and if $s < p < t$ then $p \in J$.

Let J be an interval, a its infimum and b its supremum:

$$a = \inf J, \quad \text{and} \quad b = \sup J.$$

Consider any point p strictly between a and b. Since $a < p$, the point p is not a lower bound and so there is a point $s \in J$ with $s < p$. Since $b > p$, the point p is not an upper bound, and so there is a point $t \in J$ with $p < t$. Thus

$$s < p < t.$$

Since $s, t \in J$ it follows that p, being between s and t, is also in J. The endpoints a and b themselves might or might not be in J. Thus we have the

Take for a starter example, the constant function

$$K(x) = 5 \qquad \text{for all } x \in \mathbb{R}.$$

We want to make sure that the official definition 6.1.1 does imply that

$$\lim_{x\to 3} K(x) = 5.$$

To check this consider any neighborhood of 3:

$$(3-\delta, 3+\delta),$$

where δ is any positive real number. Then

$$\sup_{x\in(3-\delta,3+\delta),x\neq 3} K(x) = 5$$

because the set of values $K(x)$ is just $\{5\}$, and also

$$\inf_{x\in(3-\delta,3+\delta),x\neq 3} K(x) = 5.$$

Thus the only value that lies between the sup and the inf is 5 itself, and hence

$$\lim_{x\to 3} K(x) = 5.$$

Now let us move to the function

$$f(x) = x \quad \text{for all } x \in \mathbb{R}.$$

We would like to make sure that Definition 6.1.1 does imply that $\lim_{x\to 6} f(x)$ is 6. Consider the neighborhood

$$(6-\delta, 6+\delta),$$

where δ is a positive real number. Then

$$\{f(x) : x \in (6-\delta, 6+\delta)\} = \{x : x \in (6-\delta, 6+\delta)\},$$

which is just the interval $(6-\delta, 6+\delta)$, but with the point 6 excluded. Hence its sup is $6+\delta$ and its inf is $6-\delta$. What value lies between these two no matter what δ is ? Certainly it is 6:

$$\inf_{x\in(6-\delta,6+\delta),x\neq 6} f(x) < 6 < \sup_{x\in(6-\delta,6+\delta),x\neq 6} f(x).$$

These are usually written as

$$\begin{aligned}
\lim_{x\to 0+} \frac{1}{x} &= \infty \\
\lim_{x\to 0-} \frac{1}{x} &= -\infty.
\end{aligned} \tag{6.6}$$

The limit of $1/x$ as $x \to 0$ does not exist. You can check that in any neghborhood of 0, excluding the value $x = 0$ itself, the sup of $1/x$ is ∞ whereas the inf of $1/x$ is $-\infty$, and so there is no *unique* value between these two extremes.

6.3 A function with no limits

Recall the set $\mathbb{Q}$ of all *rational numbers*:

$$\mathbb{Q} = \{\text{all rationals}\}$$

This is *dense* in $\mathbb{R}$:

every open interval in $\mathbb{R}$ contains rational points.

The same is true of the *irrationals*:

every open interval in $\mathbb{R}$ contains irrational points.

Consider the *indicator function* of $\mathbb{Q}$, taking the value 1 on rationals and 0 on irrationals:

$$1_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q}; \end{cases} \tag{6.7}$$

If you take any $p \in \mathbb{R}$ and any neigborhood U of p it is clear that

$$\sup_{x\in U, x\neq p} 1_Q(x) = 1 \quad \text{and} \quad \inf_{x\in U, x\neq p} 1_Q(x) = 0.$$

Thus there can be no *unique* value between these sups and infs, and so

$\lim_{x\to p} 1_Q(x)$ *does not exist for any* $p \in \mathbb{R}$.

This means that *all the values of f on the neighborhood U of 3, excluding $f(3)$ itself, are* > 4. This is exactly what we had conjectured based on common sense intuition about limits.

If you look over the preceding discussion you see that what makes the argument work is simply that 4 is a value that is $<$ than the limit 5. Thus what we have really proved is this:

Proposition 7.1.1 *Suppose f is a function on some subset $S \subset \mathbb{R}$, and*

$$L = \lim_{x \to p} f(x),$$

where $p \in \mathbb{R}^*$. If b is any value below L, that is $b < L$ then there is a neighborhood U of p on which

$$f(x) > b \text{ for all } x \in U \cap S \text{ except possibly for } x = p.$$

We have had to write $x \in U \cap S$, and not just $x \in U$, because $f(x)$ might not be defined for all x in U.

It is important not to get bogged down in the notation used: keep in mind the essense of the idea. What we are saying is in ordinary rough and ready language, if $f(x) \to L$ as $x \to p$ then the values $f(x)$ lie *above* b when x is near p (but not p itself), for any given value $b < L$.

Of course, we can do the same for values above the limit:

Proposition 7.1.2 *Suppose f is a function on some subset $S \subset \mathbb{R}$, and*

$$L = \lim_{x \to p} f(x),$$

where $p \in \mathbb{R}^*$. If u is any value $> L$ then there is a neighborhood U of p on which

$$f(x) < u \text{ for all } x \in U \cap S \text{ except possibly for } x = p.$$

We can even put thee two observations together:

Proposition 7.1.3 *Suppose f is a function on some subset $S \subset \mathbb{R}$, and*

$$L = \lim_{x \to p} f(x),$$

where $p \in \mathbb{R}^*$. If u is any value $> L$ and b is any value $< L$ then there is a neighborhood U of p on which

$$b < f(x) < u \text{ for all } x \in U \cap S \text{ except possibly for } x = p.$$

If we just do the ratio of the limits we end up with 0/0, and this is just a case where *the preceding result cannot be applied*. Thus we need to be less lazy and observe that

$$\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x+3,$$

from which it is clear that

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = 6.$$

7.4 Limits by comparing

Sometimes we can find the limit of a function by comparing it with other functions that are easier to understand.

The so called 'squeeze theorem' is a case of this. Suppose f, g, and h are functions on a set $S \subset \mathbb{R}$ and $p \in \mathbb{R}^*$ is such that

$$f(x) \le h(x) \le g(x) \tag{7.13}$$

for all x in S that lie in some neighborhood U of p, excluding $x = p$. Assume that $\lim_{x\to p} f(x)$ and $\lim_{x\to p} g(x)$ exist and are equal:

$$L = \lim_{x\to p} f(x) = \lim_{x\to p} g(x).$$

Then $h(x)$, squeezed in between $f(x)$ and $g(x)$, is forced to also approach the same limit L.

Here is a formal statement and proof:

Proposition 7.4.1 *Suppose f, g, and h are functions on a set $S \subset \mathbb{R}$, and $p \in \mathbb{R}^*$ is such that*

$$f(x) \le h(x) \le g(x) \tag{7.14}$$

for all x in S that lie in some neighborhood of p, excluding $x = p$. Assume also that $\lim_{x\to p} f(x)$ and $\lim_{x\to p} g(x)$ exist and are equal:

$$L = \lim_{x\to p} f(x) = \lim_{x\to p} g(x).$$

Then $\lim_{x\to p} h(x)$ exists and is equal to L.

Regardless of how an angle might be measured, the geometric meanings of sin, cos and tan of an acute angle are illustrated in the classical diagram shown in Figure 8.3. If the angle is specified by a pair of rays R_1 and R_2, initiating from a vertex C, we draw a circle, with center C, and take the radius to be the unit of length. The 'semichord' from R_2 to R_1 is the segment, perpendicular to R_1, that runs from the point Q where R_2 cuts the circle to a point on R_1. The length of the 'semi-chord' is the sin of the angle. The cos of the angle is the distance from the vertex C to the semi-chord. Then tan of the angle is the length of the segment tangent to the circle at Q to a point on R_1.

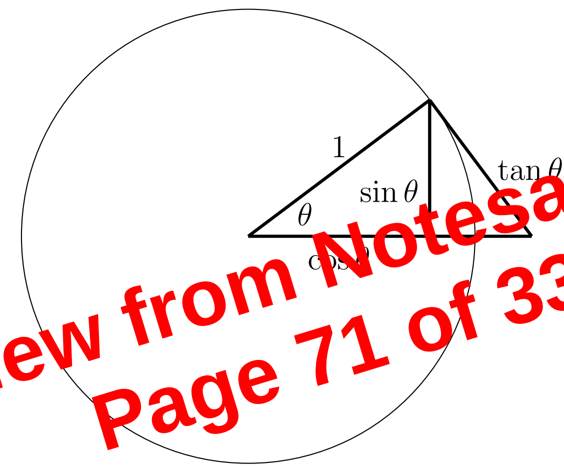

Figure 8.3: Classical definitions of sin, cos, and tan

The more cluttered Figure 8.4 provides more concrete formulas and also relates visually to the measurement of the angle θ in terms of the area of the sectorial region it cuts out of the circle.

The line through P perpendicular to CQ intersects the line CQ at a point B. Let

$$\begin{aligned} x &= CB \\ y &= QB. \end{aligned} \tag{8.1}$$

Here we take x to be *negative* if B is on the opposite side of C from P. We take y to be negative if $\theta > \pi$.

8.3 Reciprocals of sin, cos, and tan

The reciprocals of sin, cos and tan also have names:

$$\begin{aligned} \csc\theta &= \frac{1}{\sin\theta} \\ \sec\theta &= \frac{1}{\cos\theta} \\ \cot\theta &= \frac{1}{\tan\theta} \end{aligned} \tag{8.10}$$

whenever these reciprocals are meaningful (for instance, $\csc 0$ and $\sec(\pi/2)$ undefined).

8.4 Identities

If one angle of a right-angled triangle is θ then the other is $\pi/2 - \theta$. This leads to the following identities:

$$\begin{aligned} \sin\left(\frac{\pi}{2} - \theta\right) &= \cos\theta \\ \cos\left(\frac{\pi}{2} - \theta\right) &= \sin\theta \\ \tan\left(\frac{\pi}{2} - \theta\right) &= \cot\theta. \end{aligned} \tag{8.11}$$

When an angle is replaced by its negative, it changes the sign of sin and tan but not of cos:

$$\begin{aligned} \sin(-a) &= -\sin a; \\ \cos(-a) &= \cos a; \\ \tan(-a) &= -\tan a, \end{aligned} \tag{8.12}$$

with the last holding if the tan values exist.

Pythagoras' theorem implies the enormously useful identity

$$\sin^2 a + \cos^2 a = 1, \tag{8.13}$$

for all $a \in \mathbb{R}$. Using this we can work out the value of sin, at least up to sign, from the value of cos:

$$\sin a = \pm\sqrt{1 - \cos^2 a}. \tag{8.14}$$

Chapter 9

Continuity

Continuous functions are functions that respect topological structure. They are also the easiest to work with in and therefore most suitable in applications.

9.1 Continuity at a point

A function f on a set $S \subset \mathbb{R}$ is said to be *continuous* at a point $p \in S$ if $f(x)$ approaches its actual value $f(p)$ when x approaches p:

$$\text{if } \lim_{x\to p} f(x) = f(p) \text{ we say } f \text{ is continuous at } p.$$

In case p is an isolated point of S we cannot work with $\lim_{x\to p} f(x)$, but surely there is no reason to view f as being not continuous at such a point. So we also say that f is continuous at p if p is an isolated point of S.

Here is a cleaner definition of continuity at p:

Definition 9.1.1 *A function f defined on a set $S \subset \mathbb{R}$ is said to be continuous at a point $p \in S$ if for every neighborhood W of $f(p)$ there is a neighborhood U of p such that*

$$f(x) \in W \textit{ for all } x \in U.$$

9.2 Discontinuities

Sometimes a function is *discontinuous* (that is, not continuous) at a point p because the value $f(p)$ is, for whatever reason, not equal to $\lim_{x\to p} f(x)$ even where δ is any positive real number. Since s is an upper bound of S, any point p of S strictly to the right of s (that is, $p > s$) is not in S, and so

$$f(p) > t,$$

for such $p \in [a, b]$. Then

$$\sup_{x \in U, x \neq s} f(x) > t$$

Since s is the *least* upper bound of S, any point $p \in U$ for which $p < s$ is *not* an upper bound of S and so there is some $q \in S$ with $q > p$. Of course $q \leq s$, since s is an upper bound of S. Hence q, lying between p and s, is in the neighborhood U. Since $q \in S$ we have

$$f(q) < t.$$

This shows that the inf of f over U, even excluding the point s, is $< t$:

$$\inf_{x \in U, x \neq s} f(x) < t$$

Thus t satisfies:

$$\inf_{x \in U, x \neq p} f(x) < t < \sup_{x \in U, x \neq s} f(x)$$

for every neighborhood U of p. Since f is given to be continuous at s we know that

$$f(s) = \lim_{x \to s} f(x).$$

Hence t must be $f(s)$. $\boxed{\text{QED}}$

10.3 Intermediate Value Theorem: a second formulation

Here is another formulation of the intermediate value theorem:

Theorem 10.3.1 *If f is continuous on an interval J then the image*

$$f(J) \stackrel{\text{def}}{=} \{f(x) : x \in J\}$$

is also an interval.

Chapter 12

Maxima and Minima

A fundamental feature of continuous functions is that they *attain* maximum and minimum values on certain types of sets such as closed intervals $[a, b]$, for $a, b \in \mathbb{R}$ with $a < b$.

12.1 Maxima and Minima

The completeness property of the real line has another big consequence for continuous functions: if f is continuous on the interval $[a, b]$ then $f(x)$ actually attains a maximum value at some point and a minimum value on the interval $[a, b]$.

Theorem 12.1.1 *Let f be a continuous function on $[a, b]$, where $a, b \in \mathbb{R}$ with $a < b$. Then there exist $c, d \in [a, b]$ such that*

$$\begin{aligned} f(c) &= \inf_{x\in[a,b]} f(x) \\ f(d) &= \sup_{x\in[a,b]} f(x). \end{aligned} \tag{12.1}$$

Before proceeding to logical reasoning here is our strategy for finding a point where f reaches the value

$$M = \sup_{x\in[a,b]} f(x).$$

Let us follow a point t, starting at a and moving to the right towards b and keep track of the 'running supremum'

$$S_f(t) = \sup_{x\in[a,t]} f(x).$$

Hence M satisfies

$$\inf_{x \in U \cap [a,b]} f(x) \leq M \leq \sup_{x \in U \cap [a,b]} f(x),$$

wth the second $\leq$ being actually an equality. This is true for *any* neighborhood U of d_*. Therefore, by our definition of limit,

$$\lim_{x \to d_*} f(x) = M.$$

But f is continuous at d_*. Hence

$$f(d_*) = M,$$

and we are done.

Lastly suppose $d_* = b$. Then taking any $q \in [a,b]$ with$q < b$, we know that q is *not* an upper bound of B_M (for $d_* = b$ is the *least* upper bound of B_M). So there is a $p > q$ in $[a,b]$ which is in B_M, and this means $\sup_{x \in [a,p]} f(x) < M$. Therefore also

$$\sup_{x \in [a,q]} f(x) < M.$$

But since $\sup_{x \in [a,b]} f(x)$ is M, we must have $\sup_{x \in (q,b]} f(x) = M$. Thus the supremum of f over every neighborhood of d_* (which is b) is M. Then by the arhument used in the previous paragraph it follows again that $f(d_*) = M$.

The result for $\inf_{x \in [a,b]} f(x)$ is obtained similarly or just applying the result for sup to the function $-f$ instead of f. $\boxed{\text{QED}}$

The preceding heavily used result works for functions defined on closed intervals $[a,b]$, with $a, b \in \mathbb{R}$. But what of functions defined on other types of intervals? For example, for the function

$$\frac{1}{x} \qquad \text{for } x \in (0, \infty)$$

it is clear that the function is trying to reach its supremum ∞ at the left endpoint 0 and its infimum 0 at the right endpoint ∞. Figure 12.1 shows the graph of the function given on $(0, \infty)$ by $x^2 + \frac{2}{x} - 2$. The function has sup equal to ∞, which is the value it is trying to reach at both endpoints 0 and ∞ of the interval $(0, \infty)$; the inf occurs at $x = 1$ and the corresponding minimum value is $1^2 + \frac{2}{1} - 2 = 1$.

13.2 Derivative

Consider a function f defined on a set $S \subset \mathbb{R}$ and let p be a point of S that is not an isolated point. The *derivative* of f *at* p is defined to be:

$$f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p}. \tag{13.4}$$

Thus *the derivative $f'(p)$ is the slope of the tangent to the graph $y = f(x)$ at the point $\big(p, f(p)\big)$.* Of course, if the graph fails to have a tangent line then it fails to have a derivative.

Let us look at some simple examples. First consider the constant function K whose value everywhere is 5:

$$K(x) = 5 \qquad \text{for all } x.$$

Common sense tells us that the slope of this is 0. We can check this readily from the official definition

$$\lim_{x \to p} \frac{K(x) - K(p)}{x - p} = \lim_{x \to p} \frac{0}{x - p} = \lim_{x \to p} 0 = 0.$$

We can elevate this observation slightly by observing that we don't need K to be equal to 5 *everywhere*, but just on a neighborhood of p.

If the function f is constant near p, then $f(x) = f(p)$, for x in a neighborhood of p, and so the derivative $f'(p)$ is 0. This just says that the graph is flat. Thus,

> *If a function is constant on a neighborhood of a point p then the derivative of the function at p is 0.*

Next, consider the function

$$g(x) = x \qquad \text{for all } x \in \mathbb{R}.$$

Then for any real number p we have

$$\lim_{x \to p} \frac{g(x) - g(p)}{x - p} = \lim_{x \to p} \frac{x - p}{x - p} = \lim_{x \to p} 1 = 1.$$

Hence the slope of

$$y = x$$

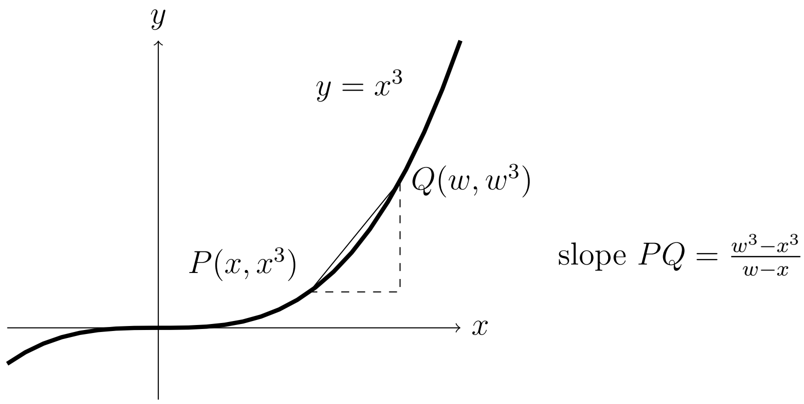

Figure 13.3: Secant segment for $y = x^3$ at $P(x, x^3)$.

13.5 Derivative of x^3

Let us do the calculation of the derivative for the function $f(x) = x^3$. Following the method used for $y = x^2$ we have first the picture

We can see that

$$\text{slope of } PQ = \frac{w^3 - x^3}{w - x}.$$

Letting $Q \to P$ makes the secant line PQ approach the tangent line at P in the limit. The slope of the tangent at P is then

$$\text{slope of tangent at } P = \lim_{w \to x} \frac{w^3 - x^3}{w - x}.$$

This just the derivative at x:

$$\begin{aligned}
\frac{dx^3}{dx} &= \lim_{w \to x} \frac{w^3 - x^3}{w - x} \\
&= \lim_{w \to x} \frac{(w - x)(w^2 + wx + x^2)}{w - x} \\
&\qquad \text{(using } A^3 - B^3 = (A - B)(A^2 + AB + B^2)) \\
&= \lim_{w \to x} (w^2 + wx + x^2) \\
&= x^2 + x^2 + x^2 \\
&= 3x^2
\end{aligned} \tag{13.9}$$

We can calculate its derivative:

$$
\begin{aligned}
\frac{d(1/x^k)}{dx} &= \lim_{w\to x} \frac{(1/w^k) - (1/x^k)}{w-x} \\
&= \lim_{w\to x} \frac{(x^k - w^k)/(x^k w^k)}{w-x} \\
&\qquad \text{(using } \tfrac{1}{A} - \tfrac{1}{B} = \tfrac{B-A}{AB}\text{)} \\
&= \lim_{w\to x} \frac{x^k - w^k}{x^k w^k (w-x)} \\
&= \lim_{w\to x} (-1) \cdot \frac{w^k - x^k}{w-x} \cdot \frac{1}{x^k w^k} \\
&= (-1) \cdot k x^{k-1} \cdot \frac{1}{x^{2k}},
\end{aligned}
\tag{13.12}
$$

where in the last step we used the derivative of x^k:

$$\lim_{w\to x} \frac{w^k - x^k}{w-x} = kx^{k-1}.$$

Thus

$$\frac{d(1/x^k)}{dx} = -k\frac{1}{x^{2k-k+1}} = -k\frac{1}{x^{k+1}}. \tag{13.13}$$

Writing n for $-k$ this reads

$$\frac{dx^n}{dx} = nx^{n-1},$$

correct again, even though n is now a negative integer.

13.9 Derivative of $x^{1/2} = \sqrt{x}$

Consider the function

$$s(x) = \sqrt{x} = x^{1/2}$$

defined on all $x \geq 0$. Consider any $p \geq 0$. Then the derivative of this function at p is the slope of the tangent at $P(p, \sqrt{p})$ to the graph $y = \sqrt{x}$, and we know that

$$\text{slope of tangent at } P = \lim_{Q\to P} (\text{slope of } PQ).$$

If $F(x)$ approaches a finite limit $F(\infty)$, as $x \to \infty$, then $F'(p)$ is 0, which conforms to intuition: the tangent line at $x = \infty$ is the 'horizontal' line $y = F(\infty)$.

Chapter 14

Derivatives of Trigonometric Functions

In this chapter we work out the derivative of sin, cos, and tan, by using their algebraic properties and the fundamental limits

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \quad\text{and}\quad \lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1.$$

14.1 Derivative of sin is cos

The derivative of the funtion sin at $x \in \mathbb{R}$ is the slope of the graph of

$$y = \sin x$$

at the point $P(x, \sin x)$. Thus it is:

$$\sin' x = \lim_{w\to x} (\text{slope of } PQ),$$

where Q is the point $(w, \sin w)$.

Now the slope of PQ is

$$\text{slope of } PQ \;=\frac{\sin w - \sin x}{w - x}.$$

We have then

$$\sin' x = \lim_{w\to x}\frac{\sin w - \sin x}{w - x}.$$

Thus

$$x_0 = \frac{(L+W) - \sqrt{L^2+W^2-LW}}{6}$$

is in the interior of $[0, W/2]$. Clearly this choice of x must produce the maximum value of the volume, for the value of $V(x)$ at the endpoints $x = 0$ and $x = W/2$ is 0.

Thus the maximum volume is

$$V(x_0) = x_0(L-2x_0)(W-2x_0)$$

After a long calculation this works out to

$$\frac{1}{54}\left[(L+W)(5LW-2L^2-2W^2) + 2(L^2+W^2-LW)\sqrt{L^2+W^2-LW}\right].$$

If we start with a square, for which $L = W$, this simplifies to

$$\frac{2}{27}L^3,$$

with x_0 being $L/6$.

Exercises on Maxima and Minima

1. Find the maximum and minimum values of x^2 for $x \in [-1, 2]$.

2. Find the maximum and minimum values of

$$x(6-x)(3-x)$$

   for $x \in [0, 2]$.

3. A wire of length 12 units is bent to form an isosceles triangle. What should the lengths of the sides of the triangle be to make its area maximum?

4. A piece of wire is bent into a rectangle of maximum area. Show that this maximal area rectangle is a square.

5. A piece of wire of length L is cut into pieces of length x and $L - x$ (including the possibilty that x is 0 or L), and each piece is bent into a circle. What is the value of x which would make the total area enclosed by the pieces maximum, and what is the value of x which would make this area minimum?

6. Here are some practice problems on straight lines and distances:

   (i) Work out the distance from $(1, 2)$ to the line $3x = 4y + 5$

   (ii) Work out the distance from $(2, -2)$ to the line $4x - 3y - 5 = 0$.

   (iii) Find the point P_0 on the line L, with equation $3x + 4y - 7 = 0$, closest to the point $(0, 3)$. What is the angle between P_0P and the line L?

   (iv) Let P_0 be the point on the line L, with equation $3x+4y-11 = 0$, closest to the point $P(1, 3)$. What is the slope of the line P_oP?

   (v) Let P_0 be the point on the line L, with equation $3x+4y-11 = 0$, closest to the point $P(1, 3)$. Find the equation of the line through P and P_0.

7. Prove the inequality
$$\frac{x^3}{3} + \frac{k^{3/2}}{3/2} \geq kx, \tag{20.18}$$
for all $x, k \in (0, \infty)$. Explain when $\geq$ is $=$. [Hint: Show that, for any fixed value $k \in (0, \infty)$, the maximum value of
$$\Phi(x) = kx - \frac{x^3}{3} \quad \text{for } x \in (0, \infty)$$
is $\frac{k^{3/2}}{3/2}$. Note that $\Phi(0) = 0$ and $\lim_{x\to\infty} \Phi(x) = -\infty$; so you have to find a point $p \in (0, \infty)$ where $\Phi'(p)$ is 0 and compare the value $\Phi(p)$ with $\Phi(0)$ and choose the larger.]

8. Prove the inequality
$$x^6 + 5k^{6/5} \geq 6kx, \tag{20.19}$$
for all $x, k \in (0, \infty)$. Now show that
$$x^6 + 5y^6 \geq 6y^5x,$$
for all $x, y \in (0, \infty)$.

<u>Proof</u>. This follows directly from the definition of the derivative:

$$f'(p) = \lim_{x\to p} \frac{f(x) - f(p)}{x - p}.$$

If f is increasing then $f(x) \geq f(p)$ when $x > p$ (thus $x - p > 0$) in S and $f(x) \leq f(p)$ when $x < p$ (thus $x - p < 0$) in S. Hence the ratio $\frac{f(x)-f(p)}{x-p}$ is ≥ 0, and so the limit $f'(p)$ is also ≥ 0.

If f is decreasing then

$$\frac{f(x) - f(p)}{x - p} \leq 0$$

both when $x > p$ and when $x < p$, with $x \in S$. Hence in this case $f'(p) \leq 0$. $\boxed{\text{QED}}$

The following is a much sharper result going in the other direction:

Proposition 23.1.2 *Let f be a function on a set $S \subset \mathbb{R}$, and p a point in S where $f'(p)$ exists and is positive, that is*

$$f'(p) > 0.$$

Then there is a neighborhood U of p such that the $f(x) > f(p)$ for $x \in U \cap S$ to the right of p and $f(x) < f(p)$ for $x \in U \cap S$ to the left of p:

$$\begin{aligned} &f(x) > f(p) \text{ for all } x \in U \cap S \text{ for which } x > p \\ &f(x) < f(p) \text{ for all } x \in U \cap S \text{ for which } x < p \end{aligned} \tag{23.1}$$

Thus, roughly put, if the slope of $y = f(x)$ is > 0 at a point p then just to the *right* of p the values of f are *higher* than $f(p)$ and just to the *left* of p the values of f are *lower* than $f(p)$.

<u>Proof</u>. Recall the definition of $f'(p)$:

$$f'(p) = \lim_{x\to p} \frac{f(x) - f(p)}{x - p}.$$

If this is > 0 then the ratio

$$\frac{f(x) - f(p)}{x - p}$$

23.2 Negative derivative and decreasing nature

The results of the preceding section can be run analogously for functions with downward pointing slope.

Proposition 23.2.1 *Let f be a function on a set $S \subset \mathbb{R}$, and p a point in S where $f'(p)$ exists and is negative, that is*

$$f'(p) < 0.$$

Then there is a neighborhood U of p such that the $f(x) < f(p)$ for $x \in U \cap S$ to the right of p and $f(x) > f(p)$ for $x \in U \cap S$ to the left of p:

$$\begin{aligned} &f(x) < f(p) \text{ for all } x \in U \cap S \text{ for which } x > p \\ &f(x) > f(p) \text{ for all } x \in U \cap S \text{ for which } x < p \end{aligned} \tag{23.2}$$

If f slopes downward along an interval then it is decreasing:

Proposition 23.2.2 *If f is defined on an interval $[a, b]$, where $a, b \in \mathbb{R}$ with $a < b$, and if $f'(p)$ exists and is negative, that is < 0, for all $p \in [a, b]$ then f is strictly decreasing on $[a, b]$ in the sense that:*

$$f(x_1) > f(x_2) \qquad \text{for all } x_1, x_2 \in [a, b] \text{ with } x_1 < x_2.$$

If f' is assumed to be ≤ 0 on $[a, b]$ then the conclusion is $f(x_1) \geq f(x_2)$.

23.3 Zero slope and constant functions

Clearly a constant function has zero slope: the derivative of a constant function is 0 wherever defined. One can run this also in the converse direction, but with just a bit of care.

Consider a function G that is defined on a domain consisting of two separated intervals, on each of which it is constant:

$$G(x) = \begin{cases} 1 & \text{if } x \in (0, 1); \\ 4 & \text{if } x \in (8, 9). \end{cases}$$

Then clearly

$$G'(p) = 0 \quad \text{for all } p \text{ in the domain of } G,$$

and yet G is, of course, not constant. On the other hand it is also clear that G really is constant, separately on each interval on which it is defined.

Proposition 23.3.1 *Suppose f is a function on an interval $[a, b]$, where $a, b \in \mathbb{R}$ with $a < b$, and $f'(p) = 0$ for all $p \in [a, b]$. Then f is constant on $[a, b]$. If f is defined on an open interval (a, b) and f' is 0 on (a, b) then f is constant on (a, b).*

One can tinker with this as usual. It is not necessary (for the case of $[a, b]$) to assume that $f'(a)$ and $f'(b)$ to exist; it suffices to assume that f is continuous at a and at b.
Proof. Consider any $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. Then by the mean value theorem

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

for some $c \in (x_1, x_2)$. So if f' is 0 everywhere it follows that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0,$$

and so

$$f(x_2) - f(x_1) = 0,$$

which means $f(x_1) = f(x_2)$. Thus the values of f at any two different points are equal; that is, f is constant. $\boxed{\text{QED}}$

Note that we have been referring to 'an' inverse function. For $y = x^2$ another choice of inverse is given by the other 'branch' of square root:

$$x = -\sqrt{y}.$$

Things could be really made messy by choosing an inverse function that switches wildly back and forth between the branches $\sqrt{y}$ and $-\sqrt{y}$. This just means that we need to exercise some care about choosing a specific well-behaved branch as an inverse functions.

24.1 Inverses and Derivatives

Suppose f is a function on an interval U such that $f'(x)$ exists for every $x \in U$ and is positive, that is $f'(x) > 0$ (alternatively we could assume that $f' < 0$ everywhere on U). Let V denote the *range* of f:

$$V = f(U) = \{f(x) : x \in U\}.$$

Since $f' > 0$ on U, f is a strictly increasing function and so it has a unique inverse function

$$f^{-1} : V \to \mathbb{R},$$

specified by the requirement that

$$f\big(f^{-1}(y)\big) = y \qquad \text{for all } y \in V.$$

Alternatively,

$$f^{-1}\big(f(x)\big) = x \qquad \text{for all } x \in U.$$

Proposition 24.1.1 *Suppose f is a function defined on an interval U, such that $f'(x)$ exists and is ≥ 0 for all $x \in U$, being equal to 0 at most at finitely many points. Then $\big(f^{-1}\big)'(y)$ exists for all $y \in V$, the range of f, and*

$$\big(f^{-1}\big)'(y) = \frac{1}{f'(x)} \tag{24.1}$$

where $x = f^{-1}(y)$; in (26.39) we take the right side $1/f'(x)$ to be 0 in case $f'(x)$ is ∞, and ∞ if $f'(x)$ is 0.

Proof of Proposition 25.1.1 . Consider any $x \in U$ to the right of p, that is $x > p$; the mean value theorem (Theorem 22.2.1) says that

$$f(x) - f(p) = (x - p)f'(c) \quad \text{for some } c \in (p, x).$$

If f' is ≥ 0 on U to the right of p then $f'(c) \geq 0$, and so we see that

$$f(x) - f(p) \geq 0 \quad \text{for } x \in U, \text{ with } x > p.$$

Thus

$$f(x) \geq f(p) \quad \text{for } x \in U, \text{ with } x > p.$$

On the other hand, taking $x < p$ but inside U we have, again by the mean value theorem,

$$f(x) - f(p) = (x - p)f'(c) \quad \text{for some } c \in (x, p),$$

but observe now that $x - p < 0$ and $f'(c)$ is given to be ≤ 0 (for c is to the left of p); hence

$$f(x) - f(p) \geq 0 \quad \text{for } x \in U, \text{ with } x < p.$$

Thus,

$$f(x) \geq f(p) \quad \text{for } x \in U, \text{ with } x < p.$$

We have shown that $f(x)$ is $\geq f(p)$ for all $x \in U$, both those to the left of p and those to the right of p. This means that f has a local minimum at p. $\boxed{\text{QED}}$

By a closely similar argument we obtain the analogous result for local maxima:

Proposition 25.1.2 *Suppose f is defined and continuous on a neighborhood U of $p \in \mathbb{R}$, and the derivative f' is ≤ 0 to the right of p and ≥ 0 to the left of p; more precisely, suppose f is differentiable on U except possibly at p, and $f'(x) \leq 0$ for $x \in U$ with $x > p$ and $f'(x) \geq 0$ for $x \in U$ with $x < p$. Then p is a local maximum for f.*

Proposition 26.5.3 *For every $a, b \in \mathbb{R}$ we have*

$$\exp(a+b) = \exp(a)\exp(b). \tag{26.24}$$

Proof. Consider any $a \in \mathbb{R}$ and let F be the function

$$F(x) = \exp(a+x) \qquad \text{for all } x \in \mathbb{R}.$$

Then

$$F'(x) = \exp'(a+x) \cdot 1 = \exp(a+x) = F(x).$$

Then by Proposition 26.5.2 we conclude that

$$F(x) = F(0)\exp(x) \text{ for all } x \in \mathbb{R}.$$

Observing that

$$F(0) = \exp(a),$$

we conclude that

$$F(x) = \exp(a)\exp(x) \text{ for all } x \in \mathbb{R}.$$

Recalling that $F(x)$ is $\exp(a+x)$ we are done. $\boxed{\text{QED}}$

It is useful to observe at this stage that $\exp(x)$ is strictly positive:

Proposition 26.5.4 *The function* exp *assumes only positive values:*

$$\exp(x) > 0 \qquad \textit{for all } x \in \mathbb{R}. \tag{26.25}$$

Proof. This follows from writing x as $x/2 + x/2$ and using the previous Proposition:

$$\exp(x) = \exp\left(\frac{x}{2} + \frac{x}{2}\right) = \exp(x/2)\exp(x/2) = [\exp(x/2)]^2.$$

This, being a square, is ≥ 0. Moreover, we know from Proposition 26.5.1 that $\exp(x/2)$ is not 0. Hence $\exp(x)$ is actually > 0. $\boxed{\text{QED}}$

From the exponential multiplicative property in Proposition 26.5.3 we have

$$\exp(2a) = \exp(a+a) = [\exp(a)]^2$$

and

$$\exp(3a) = \exp(2a+a) = \exp(2a)\exp(a) = [\exp(a)]^2\exp(a) = [\exp(a)]^3.$$

<u>Proof</u>. We can work with $a, b \in U$, with $a < b$ (if $a = b$ then (27.20) is an equality, both sides being $\Phi(a)$). Let A be the point $(a, \Phi(a))$ and B the point $(b, \Phi(b))$. The straight line joining A to be B has equation

$$y = L(x) = Mx + k$$

for some constants. Consider now any point $p \in [a, b]$; we can write this as

$$p = \lambda a + \mu b$$

for some $\lambda, \mu \in [0, 1]$ with $\lambda + \mu = 1$ (see (27.17)). The condition that the graph of Φ is below the graph of L is

$$\Phi(p) \leq L(p)$$

for all such p. Now

$$L(p) = L(\lambda a + \mu b) = \lambda L(a) + \mu L(b),$$

by Proposition 27.7.1. Since $y = L(x)$ passes through A and B, on the graph $y = \Phi(x)$, we have

$$L(a) = \Phi(a), \quad \text{and} \quad L(b) = \Phi(b). \tag{27.21}$$

Combining all these observations we have

$$\Phi(\lambda a + \mu b) \leq \lambda L(a) + \mu L(b) = \lambda \Phi(a) + \mu \Pi(b),$$

which establishes (27.20) as being equivalent to the convexity condition for Φ. For strict convexity, the point $\big(p, \Phi(p)\big)$ lies strictly below $(p, L(p))$, which means $\Phi(p) < L(p)$ when p is strictly between a and b. Translating from p to $\lambda a + \mu b$, and using again the equalities (27.21) we obtain the condition for strict convexity of Φ. $\boxed{\text{QED}}$

It is now easy to raise the inequality (27.20) to an inequality for convex combinations for multiples points. For example, for points $p_1, p_2, p_3 \in U$, we have

$$\begin{aligned}
\Phi(w_1p_1 + w_2p_2 + w_3p_3) &= \Phi\left(w_1p_1 + (1 - w_1)\left(\frac{w_2p_2 + w_3p_3}{1 - w_1}\right)\right) \\
&\leq w_1L(p_1) + (1 - w_1)L\left(\frac{w_2p_2 + w_3p_3}{1 - w_1}\right) \\
&= w_1L(p_1) + (1 - w_1)\left(\frac{w_2}{1 - w_1}L(p_2) + \frac{w_3}{1 - w_1}L(p_3)\right) \\
&= w_1L(p_1) + w_2L(p_2) + w_3L(p_3).
\end{aligned}$$

Exercises on Maxima/Minima, Mean Value Theorem, Convexity

1. Find the maximum value of $x^{2/x}$ for $x \in (0, \infty)$. Explain your reasoning fully and present all calculations clearly.

2. Find the distance of the point $(1, 2)$ from the line whose equation is

$$3x + 4y - 5 = 0.$$

3. Suppose f is a twice differentiable function on $[1, 5]$, with $f(1) = f(3) = f(5)$. Show that there is a point $p \in (1, 5)$ where $f''(p)$ is 0.

4. Explain briefly why

$$\log 101 - \log 100 < .01.$$

5. Prove the inequality

$$\frac{1}{\left(\frac{a+b}{2}\right)^2} \leq \frac{1}{2}\frac{1}{a^2} + \frac{1}{2}\frac{1}{b^2}$$

for any $a, b > 0$.

28.2 Proving l'Hospital's rule

The key step in proving l'Hospital

$$\lim_{x\to p}\frac{f(x)}{g(x)} = \lim_{x\to p}\frac{f('x)}{g'(x)},$$

with $f(x)$ and $g(x)$ both $\to 0$ as $x \to p$, is the observation that

$$\frac{f(x)}{g(x)} = \frac{f('c)}{g'(c)},$$

for some c between x and p; when $x \to p$, the point c also $\to p$ and this shows that the above ratios approach the same limit. This is formalized in the following version of the mean value theorem:

Proposition 28.2.1 *Suppose F and G are continuous functions on a closed interval $[a, b]$, where $a, b \in \mathbb{R}^*$ and $a < b$, with values in $\mathbb{R}$. Suppose that F and G are differentiable on (a, b), with $G'(x) \neq 0$ for all $x \in (a, b)$. Then*

$$\frac{F(b) - F(a)}{G(b) - G(a)} = \frac{F'(c)}{G'(c)} \tag{28.7}$$

for some $c \in (a, b)$.

Since G' is never 0 on (a, b) it follows by Rolle's theorem that $G(b)-G(a) \neq 0$.
Proof. Consider the function H defined on $[a, b]$ by

$$H(x) = [G(b) - G(a)]\,[F(x) - F(a)] - [F(b) - F(a)]\,[G(x) - G(a)] \quad \text{for all } x \in [a, b]. \tag{28.8}$$

This is clearly continuous on $[a, b]$ and differentiable on (a, b) with derivative given by

$$H('x) = [G(b) - G(a)]\,F'(x) - [F(b) - F(a)]\,G'(x) \tag{28.9}$$

for all $x \in (a, b)$.

Observe also that

$$H(a) = H(b) = 0.$$

The width of the k-th interval is denoted

$$\Delta x_k = x_k - x_{k-1}. \tag{29.5}$$

For a function

$$f : [a, b] \to \mathbb{R},$$

and the partition P, the *upper sum* is

$$U(f, P) = \sum_{k=1}^{N} M_k(f)\Delta x_k, \tag{29.6}$$

and the *lower sum* is

$$L(f, P) = \sum_{k=1}^{N} m_k(f)\Delta x_k, \tag{29.7}$$

where

$$\begin{aligned} M_k(f) &= \sup_{x\in[t_{k-1},t_k]} f(x) \\ m_k(f) &= \inf_{x\in[t_{k-1},t_k]} f(x). \end{aligned} \tag{29.8}$$

In the degenerate case where $b = a$, the only partition of $[a, a]$ is just the one-point set $\{a\}$, and the upper and lower sums are taken to be 0.

If there is a unique value A for which

$$L(f, P) \leq A \leq U(f, P) \tag{29.9}$$

for every partition P of $[a, b]$, then A is called the *Riemann integral* of f, and denoted

$$\int_a^b f.$$

We will refer to this simply as *the integral of f from a to b* or *over* $[a, b]$.

We say that f is *integrable* if $\int_a^b f$ exists and is finite (in $\mathbb{R}$).

The definition of the integral here is in the same spirit that of the concept of limit back in (6.1) and the concept of tangent line in (13.1).

From (29.9) we see that an approximation to $\int_a^b f($ is given by

$$\int_a^b f(x)\,dx \simeq \sum_{k=1}^{N} f(x_k^*)\Delta x_k, \tag{29.10}$$

Proposition 29.3.1 *Let $f : [a, b] \to \mathbb{R}$ be a function, where $a, b \in \mathbb{R}$ and $a \leq b$, and P and P' any partitions of $[a, b]$ with $P \subset P'$; then*

$$\begin{aligned} L(f, P) &\leq L(f, P') \\ U(f, P') &\leq U(f, P). \end{aligned} \tag{29.12}$$

This implies the following natural but strong observation:

Proposition 29.3.2 *Let $f : [a, b] \to \mathbb{R}$ be a function, where $a, b \in \mathbb{R}$ and $a \leq b$, and P and Q any partitions of $[a, b]$; then*

$$L(f, P) \leq U(f, Q). \tag{29.13}$$

Thus, every upper sum of f is $\geq$ every lower sum of f.

We have seen something similar in our study of limits back in (6.14).
Proof. Let

$$P' = P \cup Q.$$

Then P' contains both P and Q, and so by Proposition 29.3.1 we have

$$L(f, P) \leq L(f, P') \qquad \text{and} \qquad U(f, P') \leq U(f, Q).$$

Combining this with the fact that $L(f, P') \leq U(f, P')$ produces the inequality (29.13). $\boxed{\text{QED}}$

29.4 Estimating approximation error

Consider a function f on an interval $[a, b] \subset \mathbb{R}$, and let $P = \{x_0, \ldots, x_N\}$ be a partition of $[a, b]$ with

$$a = x_0 < \ldots < x_N = b.$$

We know that the integral of f, if it exists, lies between the upper sum $U(f, P)$ and the lower sum $L(f, P)$. So if $U(f, P)$ and $L(f, P)$ are close to each other then either of these sums would be a good approximation to the value of the integral. Let us find how far from each other the upper and lower sums are:

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{k=1}^{N} M_k(f)\Delta x_k - \sum_{k=1}^{N} m_K(f)\Delta x_k \\ &= \sum_{k=1}^{N} \left[M_k(f) - m_k(f)\right] \Delta x_k, \end{aligned} \tag{29.14}$$

for all $x \in [x_0, x_1]$. We can clearly take $x_1 \leq b$, as here is no need to go beyond b. Thus

$$\sup_{x\in[x_0,x_1]} f(x) \leq f(x_0) + \frac{\epsilon}{4}$$

and

$$\inf_{x\in[x_0,x_1]} f(x) \geq f(x_0) - \frac{\epsilon}{4}.$$

These conditions imply that the fluctuation of f over $[x_0, x_1]$ is $\leq \epsilon/2$, which is, of course, $< \epsilon$.

Now we can start at x_1, if it isn't already b, and produce a point $x_2 > x_1$ for which

$$f(x_1) - \frac{\epsilon}{4} < f(x) < f(x_1) + \frac{\epsilon}{4}$$

for all $x \in [x_1, x_2]$. Again, we can take $x_2 \leq b$. It might seem that in this way we could produce the desired partition P. But there could be a problem: the process might continue infinitely without reaching b. Fortunately, we can show that this will not happen.

Suppose s is the supremum of all $t \in [a, b]$ such that $[a, t]$ has a partition $P_0 = \{x_0, \ldots, x_K\}$ for which the fluctuations of f over every interval of the partition is $< \epsilon$. Note that $s > a$. By continuity of f at s there is an interval (p, q), centered at s, such that the fluctuation of f over $(p, q) \cap [a, b]$ is $< \epsilon$. Pick any point $t \in (p, s)$, with $t > a$; then since $t < s$ the definition of s implies that there is a partition

$$P_0 = \{x_0, \ldots, x_K\}$$

of $[a, t]$ such that the fluctuation of f over each interval $[x_{j-1}, x_j]$ is $< \epsilon$. Now pick any point $r \in [s, q) \cap [a, b]$ and set

$$x_{K+1} = r.$$

Since the fluctuation of f over (p, q) is $< \epsilon$, the fluctuation of f over the subinterval $[t, r]$ is $< \epsilon$. Thus we have produced a point r, which is $\geq s$, such that there is a partition $P = \{x_0, \ldots, x_{K+1}\}$ of $[a, r]$ for which the fluctuations of f are all $< \epsilon$. To avoid a contradiction with the definition of s, we must have $s = b$ (for otherwise, if $s < b$, we could have chosen r to be $> s$) and the partition P has the desired fluctuation property. $\boxed{\text{QED}}$

Now we can prove Theorem 29.5.1.

This makes the upper sum large:

$$U(1_{\mathbb{Q}}, P) = 1 \cdot \Delta x_1 + \ldots + 1 \cdot \Delta x_N = b - a,$$

and the lower sum small:

$$L(1_{\mathbb{Q}}, P) = 0 \cdot \Delta x_1 + \ldots + 0 \cdot \Delta x_N = 0.$$

There certainly are many real numbers lying between 0 and $b - a$, and so there is no unique such choice. Hence,

$$\int_a^b 1_{\mathbb{Q}} \quad \text{does not exist.}$$

29.7 Basic properties of the integral

Integration of a larger function produces a larger number:

Proposition 29.7.1 *If f and g are functions on an interval $[a, b]$, where $a, b \in \mathbb{R}$ and $a \leq b$, for which the integrals $\int_a^b f$ and $\int_a^b g$ exist, and if $f \leq g$ then*

$$\int_a^b f \leq \int_a^b g.$$

Proof. Suppose $\int_a^b f > \int_a^b g$. Since $\int_a^b g$ is the *unique* value lying between $L(g, P)$ and $U(g, P)$ for all partitions P of $[a, b]$, there must be a partition P of $[a, b]$ such that

$$\int_a^b f > U(g, P).$$

Again, since $\int_a^b f$ is the unique value lying between all upper and lower sums for f there is a partition Q of $[a, b]$ for which

$$L(f, Q) > U(g, P). \tag{29.18}$$

Let

$$P' = P \cup Q,$$

which is a partition of $[a, b]$. Since $Q \subset P'$ and $f \leq g$ we have

$$L(f, Q) \leq L(g, Q) \leq L(g, P').$$

All of this notation has been designed to produce this notational consistency:

$$\frac{df(x)}{dx} = f'(x), \tag{30.7}$$

where on the left we now have a genuine ratio (of functions), not just a formal one.

Using equation (30.6) we can easily verify the following convenient identities:

$$\begin{aligned}
d(f+g) &= df + dg \\
dC &= 0 \qquad \text{if } C \text{ is constant} \\
d(fg) &= (df)g + f\,dg \\
d\left(\frac{f}{g}\right) &= \frac{g\,df - f\,dg}{g^2} \\
df(g(x)) &= f'(g(x))dg(x) \qquad \text{(this is from the chain rule)},
\end{aligned} \tag{30.8}$$

where f and g are differentiable functions on some common domain except that in the last identity we assume the composite $f(g(x))$ is defined on some open interval. (Note $\frac{\Phi}{f}$ means $\frac{1}{f}\Phi$ for any differential form Φ and function f.)

As an example, we have

$$d\log(\sin x^2) = \frac{1}{\sin x^2}\cos(x^2) * 2x\,dx.$$

If f is a differentiable function on an interval containing points a and b we define the integral of the differential df to be

$$\int_a^b df = f(b) - f(a). \tag{30.9}$$

For example,

$$\int_{\pi/2}^{\pi} d(\sin x) = \sin\pi - \sin(\pi/2) = 0 - 1 = -1,$$

and

$$\int_1^0 de^x = e^0 - e^1 = 1 - e,$$

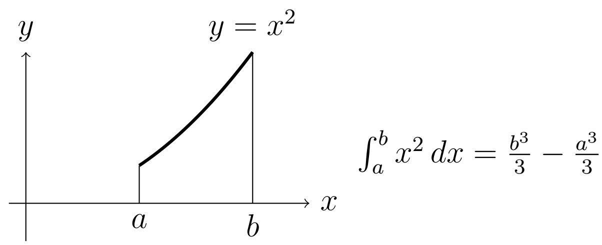

Figure 30.2: Area below $y = x^2$ for $x \in [a, b]$

$$\int_a^b x^2\,dx = \int_a^b (x^3/3)'\,dx \overset{\text{Thm.30.1.2}}{=} \left.\frac{x^3}{3}\right|_a^b = \frac{b^3}{3} - \frac{a^3}{3}.$$

Archimedes amazing determination of areas assocated with parabolas has thus been reduced to a simple routine calculation.

Next consider the area under

$$y = \sin x \qquad x \in [0, \pi].$$

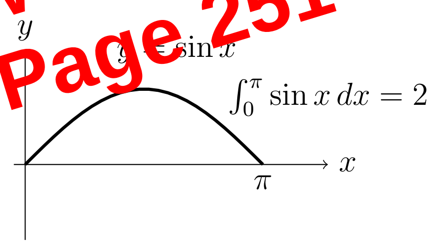

Figure 30.3: Area below $y = \sin x$ for $x \in [0, \pi]$

The area is

$$\begin{aligned}\int_0^\pi \sin x\,dx &= \int_0^\pi (-\cos x)'\,dx \\ &= -\cos x\Big|_0^\pi \\ &= (-\cos\pi) - (-\cos 0) = (-(-1)) - (-1) \\ &= 2.\end{aligned} \tag{30.13}$$

We can extract some information from this by focusing on the first inequality:

$$s_N \stackrel{\text{def}}{=} \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{N^2} \leq 1 - \frac{1}{N}. \tag{31.3}$$

This is true for all integers $N \in \{2, 3, \ldots\}$. Observe that the sequence of sums s_1, s_2, ... increases in value as additional terms are added on:

$$s_1 < s_2 < s_3 < \cdots$$

Therefore, there is a limit

$$\lim_{N\to\infty} s_N = \sup_{N\in\{2,3,\ldots\}} s_N.$$

From (31.3) we see that

$$\lim_{N\to\infty} \leq 1.$$

Thus,

$$\lim_{N\to\infty} \left[\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{N^2}\right] \leq 1.$$

This limit is displayed as an 'infinite series sum':

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Since the value of this is finite (being ≤ 1) the value of

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

is also finite, having value ≤ 2. One says that the series

$$\sum_n \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \tag{31.4}$$

converges.

The convergence of the series above can be seen in other ways, but the method using the integral $\int 1/x^2\,dx$ is useful for other similar sums as well.

The actual value of the sum (31.4) can be computed by more advanced methods; the amazing result is

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}. \tag{31.5}$$

Unlike what happened with $1/x^2$, we have an infinite area under the graph of $1/x$ over $[1,\infty)$:

$$\int_1^\infty \frac{dx}{x} \stackrel{\text{def}}{=} \lim_{t\to\infty} \int_1^t \frac{dx}{x} = \lim_{t\to\infty} \log(t) = \infty. \tag{31.8}$$

Looking back to the second inequality in (31.7) we have

$$\lim_{N\to\infty} \left[\frac{1}{1} + \frac{1}{2} + \cdots\right] \geq \lim_{N\to\infty} \log(N) = \infty,$$

and so

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots = \infty. \tag{31.9}$$

The series $\sum_{n=1}^{\infty}$ is called the *harmonic series* and the fact that the sum is ∞ is expressed by saying that series is *divergent*.

The difference between the upper sum and the integral over $[1,N]$ is

$$\sum_{k=1}^{N-1} \frac{1}{k} - \log(N).$$

It turns out that this has a finite limit as $N\to\infty$:

$$\gamma = \lim_{N\to\infty} \left[\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{N} - \log(N)\right], \tag{31.10}$$

called *Euler's constant*.

31.3 Riemann sums for x

We focus on the function given by $f(x) = x$ on $[0,1]$.

Let N be a an integer > 1. Consider the partition of $[0,1]$ given by

$$P = \left\{0, \frac{1}{N}, \ldots, \frac{N}{N}\right\}.$$

This breaks up $[0,1]$ into N intervals, each of width 1:

$$\left[0, \frac{1}{N}\right], \left[\frac{1}{N}, \frac{2}{N}\right] \ldots, \left[\frac{N-1}{N}, \frac{N}{N}\right].$$

Then

$$dy = 1\,dx$$

and so

$$\int (x+1)^5\,dx = \int y^5\,dy = \frac{1}{6}y^6 + C = \frac{1}{6}(x+1)^6 + C,$$

where C is an arbitrary constant.

The essence of the idea behind the substitution method is simple. We inspect the integral

$$\int f(x)\,dx$$

and write it in the form

$$\int F\big(p(x)\big)\,p'(x)dx,$$

for some functions F and p, and then substitute

$$y = p(x)$$

to transform the given integral as

$$\int f(x)\,dx = \int F\big(p(x)\big)\,p'(x)dx = \int F\big(p(x)\big)\,dp(x) = \int F(y)\,dy,$$

and, if al goes well, the integral $\int F(y)\,dy$ is 'simpler' than what we started with, thereby reducing $\int f(x)\,dx$ to a simpler integral. The main challenge is in identifying the functions F and p which express $f(x)$ as $F(p(x))p'(x)$.

As we will see below there are also some simple variations on this strategy. For example, it may be easier to write $f(x)$ as a constant multiple of $F(p(x))p'(x)$ or, in some cases, we can break up $f(x)$ into a sum of pieces, each of which is easier to work out separately.

Consider

$$\int (2x-5)^{3/5}\,dx.$$

We substitute

$$z = 2x - 5$$

which gives

$$dz = 2dx \qquad \text{and so} \qquad dx = \frac{1}{2}dz$$

and then the given integral has the form

$$\int xy^{2/5}\left(-\frac{1}{4}dy\right).$$

We need to replace the x in the integrand with its expression in terms of y by solving (32.1):

$$x = \frac{1}{4}(3-y).$$

Then our integral becomes

$$\int \frac{1}{4}(3-y)y^{2/5}\left(-\frac{1}{4}dy\right) = -\frac{1}{16}\int (3-y)y^{2/5}\,dy.$$

The right side looks complicated but can be worked out by breaking up into pieces:

$$\int (3-y)y^{2/5}\,dy = 3\int y^{2/5}\,dy - \int \underbrace{y\cdot y^{2/5}}_{y^{7/5}}\,dy = 3\frac{1}{\frac{2}{5}+1}y^{\frac{2}{5}+1} - \frac{1}{\frac{7}{5}+1}y^{7/5+1} + \text{constant}$$

Now putting everything together we have

$$\int x(3-4x)^{2/5}\,dx = \left(-\frac{1}{16}\right)\frac{15}{7}y^{7/5} - \left(-\frac{1}{16}\right)\frac{5}{12}y^{12/5} + C,$$

where C is an arbitrary constant and y is as in (32.1). Thus

$$\int x(3-4x)^{2/5}\,dx = -\frac{15}{112}(3-4x)^{2/5} + \frac{5}{192}(3-4x)^{12/5} + C,$$

for any arbitrary constant C.

Moving on to more complicated integrands, consider

$$\int (x^2+1)^{2/3}x\,dx$$

Observe that xdx is about the same as $d(x^2+1)$, aside from a constant multiple. The substitution is

$$y = x^2 + 1.$$

from which we have

$$
\begin{aligned}
\int \sin^2 x\, dx &= \frac{1}{2}\int [1-\cos(2x)]\, dx \\
&= \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right] + \text{constant.} \\
&= \frac{x}{2} - \frac{1}{4}\sin(2x) + \frac{1}{2} + C,
\end{aligned}
\tag{32.14}
$$

where C is an arbitrary constant. Using the formula

$$\sin(2x) = \sin x \cos x$$

we an rewrite the above integral also as:

$$\int \sin^2 x\, dx = \frac{x}{2} - \frac{1}{2}\sin x \cos x + C.$$

Similarly for $\cos^2 x$ we have

$$\cos^2 x = \cos x \cos x = \frac{1}{2}[\cos(0) + \cos(2x)] = \frac{1}{2}[1 + \cos(2x)],$$

which leads to

$$\int \cos^2 x\, dx = \frac{1}{2}\left[x + \frac{1}{2}\sin(2x)\right]. \tag{32.15}$$

We can use the sum-formula strategy multiple times. For example,

$$
\begin{aligned}
\sin(5x)\sin(3x)\cos(4x) &= \frac{1}{2}[\cos(2x) - \cos(8x)]\cos(4x) \\
&= \frac{1}{2}[\cos(2x)\cos(4x) - \cos(8x)\cos(4x)] \\
&= \frac{1}{2}\left[\frac{1}{2}[\cos(6x) + \cos(2x)] - \frac{1}{2}[\cos(12x) + \cos(4x)]\right] \\
&= \frac{1}{4}[\cos(6x) + \cos(2x) - \cos(12x) - \cos(4x)]
\end{aligned}
\tag{32.16}
$$

from which we have

$$\int \sin(5x)\sin(3x)\cos(4x)\, dx = \frac{1}{4}\left[\frac{1}{6}\sin(6x) + \frac{1}{2}\sin(2x) - \frac{1}{12}\sin(12x) - \frac{1}{4}\sin(4x)\right]$$

and so we use the subtitution

$$y = \sec x + \tan x.$$

Then, as we just observed,

$$dy = (\sec x) y \, dx,$$

and so

$$\sec x \, dx = \frac{dy}{y},$$

from which we have

$$\int \sec x \, dx = \int \frac{dy}{y} = \log y + C.$$

Thus

$$\int \sec x \, dx = \log(\sec x + \tan x) + C. \tag{32.21}$$

Again, we can use $\log|\cdot|$ if $\sec x + \tan x$ is negative.

32.4 Using trigonometric substitutions

For the integral

$$\int \frac{dx}{\sqrt{1 - x^2}}$$

the best substitution is

$$x = \sin \theta.$$

This means we are setting θ to be an inverse sin of x; for definiteness we can set

$$\theta = \sin^{-1}(x),$$

Thus,

$$\int \log x \, dx = x \log x - x + C, \tag{32.30}$$

where C is an arbitrary constant.

Sometimes the choice of U and V requires some planning ahead:

$$\begin{aligned}
\int x \log x \, dx &= \frac{1}{2} \int \log x \, d(x^2) \\
&= \frac{1}{2}(\log x)x^2 - \frac{1}{2} \int x^2 \, d(\log x) \\
&= \frac{1}{2}(\log x)x^2 - \frac{1}{2} \int x^2 \frac{dx}{x} \\
&= \frac{1}{2}(\log x)x^2 - \frac{1}{2} \int x \, dx \\
&= \frac{1}{2}(\log x)x^2 - \frac{1}{6}x^3 + C,
\end{aligned} \tag{32.31}$$

where C is an arbitrary constant.

We can apply this method to

$$\int x \sin x \, dx$$

as follows:

$$\begin{aligned}
\int x \sin x \, dx &= -\int x \, d(\cos x) \\
&= -\left[x(\cos x) - \int \cos x \, dx \right] \\
&= -x \cos x + \int \cos x \, dx \\
&= -x \cos x + \sin x + C
\end{aligned} \tag{32.32}$$

where C is an arbitrary constant.

Dividing by $A^2 + B^2$ produces, at last, the formula

$$\int e^{Ax} \sin(Bx)\,dx = e^{Ax}\left[\frac{A\sin(Bx) - B\cos(Bx)}{A^2+B^2}\right] + C, \tag{32.38}$$

where C is an arbitrary constant. We assumed A and B are both nonzero. We can check easily that the formula works even if one of these two values is 0.

Exercises on Integration by Substitution

1. Work out the following integrals using substitutions:

   (a) $\int (4-3x)^{2/3}\,dx$

   (b) $\int \sqrt{2+5x}\,dx$

   (c) $\int \frac{1}{\sqrt{2-3x}}\,dx$

   (d) $\int x(3-2x)^{4/5}\,dx$

   (e) $\int \frac{x}{(2+5x)^{3/5}}\,dx$

   (f) $\int \frac{2x+1}{\sqrt{x^2+x+5}}\,dx$

   (g) $\int e^{-x^2/2}x\,dx$

   (h) $\int \frac{\sqrt{\log(2x+5)}}{2x+5}\,dx$

   (i) $\int \frac{1}{x\log(x)\log(\log x)}\,dx$

   (j) $\int \sin(5x)\cos(2x)\,dx$

   (k) $\int \sin(5x)\sin(2x)\,dx$

   (l) $\int \cos(5x)\cos(2x)\,dx$

   (m) $\int \sin^3 x\,dx$

   (n) $\int \cos^3 x\,dx$

   (o) $\int \sin^2(5x)\,dx$

   (p) $\int \sqrt{3-6x-x^2}\,dx$

   (q) $\int \frac{1}{\sqrt{3-6x-x^2}}\,dx$

Chapter 33

Paths and Length

33.1 Paths

A *path* c in the plane $\mathbb{R}^2$ is a mapping

$$c : I \to \mathbb{R}^2 : t \mapsto c(t) = \big(xc(t), yc(t)\big), \tag{33.1}$$

where I is some interval in $\mathbb{R}$. We can think of $c(t)$ as being the *position* of a point at *time* t.

In (33.1) we are denoting the x-coordinate of a point p by $x(p)$:

$$x(p) = x\text{-coordinate of a point } p,$$

so that the x-coordinate of $c(t)$ is $x\big(c(t)\big)$, which we write briefly as

$$xc(t).$$

Similarly, the y-coordinate of a point p is

$$y(p) = y\text{-coordinate of a point } p,$$

and the y-coordinate of $c(t)$ is $y\big(c(t)\big)$, which we write briefly as $yc(t)$.

As our first example, consider

$$c(t) = (t, 2t + 1) \qquad \text{for } t \in \mathbb{R}.$$

Think of this as a moving point, whose position at clock time t is $(t, 2t + 1)$; see Figure 33.1.

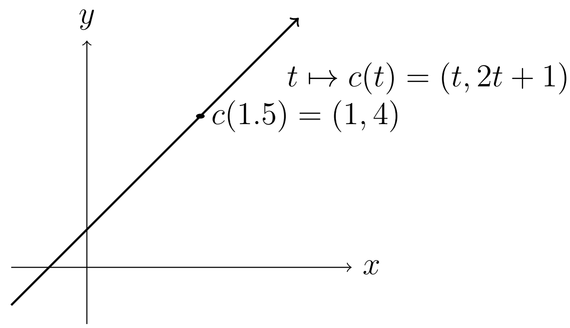

Figure 33.1: The path $c : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (t, 2t + 1)$

This is a point traveling at a uniform speed along a straight line. How fast is it traveling? We can check how fast the x and y coordinates are changing:

$$c'(t) = \big((xc)'(t), (yc)'(t)\big) = \left(\frac{dt}{dt}, \frac{d(2t+1)}{dt}\right) = (1, 2).$$

This is called the *velocity* of the path c at time t. Note that for this path the velocity is the same, being $(1, 2)$, at all times t.

Here is a *different* path that also travels along the same line, but with increasing speed:

$$\mathbb{R} \to \mathbb{R}^2 : t \mapsto c(t) = (t^2, 2t^2 + 1).$$

This is displayed in Figure 33.2.

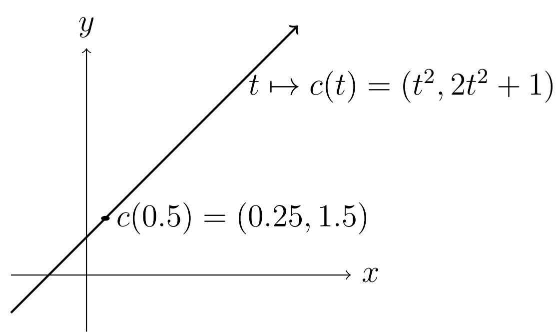

Figure 33.2: The path $c : \mathbb{R} \to \mathbb{R}^2 : t^2 \mapsto (t, 2t^2 + 1)$

The velocity of this path at time t is

$$c'(t) = (2t, 4t),$$

16.

$$\lim_{x\to\infty}\sqrt{x+2}\left[\sqrt{x+1}-\sqrt{x}\right] = \lim_{x\to\infty}\sqrt{x+2}\frac{\left[\sqrt{x+1}-\sqrt{x}\right]\left[\sqrt{x+1}+\sqrt{x}\right]}{\left[\sqrt{x+1}+\sqrt{x}\right]}$$
$$= \lim_{x\to\infty}\sqrt{x+2}\frac{x+1\,-\,x}{\sqrt{x+1}+\sqrt{x}}$$
$$= \lim_{x\to\infty}\sqrt{x+2}\frac{1}{\sqrt{x+1}+\sqrt{x}}$$
$$= \lim_{x\to\infty}\sqrt{x(1+2/x)}\frac{1}{\sqrt{x(1+1/x)}+\sqrt{x}}$$
$$= \lim_{x\to\infty}\sqrt{x}\sqrt{1+2/x}\frac{1}{\sqrt{x}\sqrt{1+1/x}+\sqrt{x}}$$
$$= \lim_{x\to\infty}\sqrt{x}\sqrt{1+2/x}\frac{1}{\sqrt{x}\left(\sqrt{1+1/x}+1\right)}$$
$$= \lim_{x\to\infty}\sqrt{1+2/x}\frac{1}{\sqrt{1+1/x}+1}$$
$$= 1\cdot\frac{1}{1+1} = \frac{1}{2}.$$

17. $\lim_{\theta\to 0}\frac{\sin(\theta^2)}{\theta^2} = \lim_{y\to 0}\frac{\sin y}{y} = 1$, on setting

$$y = \theta^2$$

and noting that $y \to 0$ as $\theta \to 0$.

18. $\lim_{\theta\to 0}\frac{\sin^2(\theta)}{\theta^2} = \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^2 = 1^2 = 1$

19. $\lim_{\theta\to\pi/6}\frac{\sin(\theta-\pi/6)}{\theta-\pi/6} = \lim_{x\to 0}\frac{\sin x}{x} = 1$, on setting

$$x = \theta - \pi/6$$

and noting that this $\to 0$ as $\theta \to \pi/6$.

20. $\lim_{x\to 0} x^2 1_{\mathbb{Q}}(x) = 0$ from the 'squeeze' theorem on using

$$0 \le \left|x^2 1_{\mathbb{Q}}(x)\right| \le x^2 \to 0 \qquad \text{as } x \to 0.$$

21. $\lim_{x\to 0} x(1-x)1_{\mathbb{Q}}(x) = 0$ from the 'squeeze' theorem on using

$$|x(1-x)1_{\mathbb{Q}}(x)| \le |x(1-x)| \to 0 \qquad \text{as } x \to 0.$$

22. $\lim_{x\to 1} x(1-x)1_{\mathbb{Q}}(x) = 0$ from the 'squeeze' theorem on using

$$|x(1-x)1_{\mathbb{Q}}(x)| \le |x(1-x)| \to 0 \qquad \text{as } x \to 1.$$

23. Explain why $\lim_{x\to 3} x(x-1)1_{\mathbb{Q}}(x)$ does not exist.

Sol: Near $x = 3$ the supremum of the values $x(x-1)1_{\mathbb{Q}}(x)$ is around $3(3-1) = 6$ (actually, more than this, because if $x > 3$, with x *rational*, then $x(x-1)1_{\mathbb{Q}}(x) = x(x-1) > 3(3-1)$), whereas the inf is 0 on taking x irrational.

24. Explain why $\lim_{x\to\infty} \cos x$ does not exist.

Sol: $\cos x$ oscillates between 1 and -1 as x runs from any integer multiple of 2π (such as 0, 2π, 4π, 6π,...) and the next higher such multiple. So for any positive real number t we have

$$\sup_{x\in(t,\infty)} \cos x = \infty, \qquad \text{and} \qquad \inf_{x\in(t,\infty)} \cos x = -1.$$

$$\begin{aligned}
\int x(3-2x)^{4/5}\,dx &= \int xy^{4/5}\left(-\frac{1}{2}dy\right) \\
&= -\frac{1}{2}\int \frac{3-y}{2}y^{4/5}\,dy \quad (\text{using } x=\tfrac{3-y}{2}) \\
&= -\frac{1}{4}\int (3y^{4/5}-y^{9/5})\,dy \\
&= -\frac{1}{4}\left[3\frac{1}{\frac{4}{5}+1}y^{\frac{4}{5}+1}-\frac{1}{\frac{9}{5}+1}y^{\frac{9}{5}+1}\right]+C \\
&= -\frac{5}{12}y^{9/5}+\frac{5}{56}y^{7/5}+C \\
&= -\frac{5}{12}(3-2x)^{9/5}+\frac{5}{56}(3-2x)^{7/5}+C
\end{aligned} \tag{34.5}$$

v. $\int \frac{x}{(2+5x)^{3/5}}\,dx$

Use the substitution

$$y = 2+5x.$$

Then

$$dy = 5dx,$$

and

$$x = \frac{1}{5}(y-2).$$

xiii. $\int \cos(5x)\cos(2x)\,dx = \frac{1}{2}\left[\frac{1}{7}\sin 7x + \frac{1}{3}\sin 3x\right] + C$

xiv. $\int \sin^3 x\,dx$

$$\begin{aligned}
\sin^3 x &= \sin x \sin x \sin x \\
&= \frac{1}{2}\left[\cos 0 - \cos(2x)\right]\sin x \\
&= \frac{1}{2}\left[1 - \cos 2x\right]\sin x \\
&= \frac{1}{2}\left[\sin x - \sin x \cos 2x\right] \\
&= \frac{1}{2}\left[\sin x - \frac{1}{2}\left[\sin 3x + \sin(-x)\right]\right] \\
&= \frac{1}{2}\sin x - \frac{1}{4}\left[\sin 3x - \sin x\right] \\
&= \frac{1}{2}\sin x + \frac{1}{4}\sin x - \frac{1}{4}\sin 3x \\
&= \frac{3}{4}\sin x - \frac{1}{4}\sin 3x
\end{aligned} \tag{34.9}$$

Integration gives

$$\int \sin^3 x\,dx = -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C.$$

xv. $\int \cos^3 x\,dx$

$$\begin{aligned}
\cos^3 x &= \cos x \cos x \cos x \\
&= \frac{1}{2}\left[\cos 0 + \cos(2x)\right]\cos x \\
&= \frac{1}{2}\left[1 + \cos 2x\right]\cos x \\
&= \frac{1}{2}\left[\cos x + \cos x \cos 2x\right] \\
&= \frac{1}{2}\left[\cos x + \frac{1}{2}\left[\cos 3x + \cos(-x)\right]\right] \\
&= \frac{1}{2}\cos x + \frac{1}{4}\left[\cos 3x + \cos x\right] \\
&= \frac{3}{4}\cos x + \frac{1}{4}\cos 3x
\end{aligned} \tag{34.10}$$