$f(s) = \sum_{i=1}^{n} a_i \phi_i(s)$ $a_i \in \mathbb{C}$ $\{\phi_i(s)\}$: basis $\langle f, g\rangle = \int f^* g \, ds$: IP

$\|f\| = \sqrt{\langle f, f\rangle}$: Norm $\langle \phi_i, \phi_j \rangle = \delta_{ij}$: ON basis $a_n = \langle \phi_n, f\rangle$

$f(x) = \sum_i a_i \phi_i(x) = \sum_j b_j \psi_j(x)$ $\{\phi_i, \psi_j\}$: ON $b_j = \sum_i a_i \langle \psi_j, \phi_i \rangle$

$\{\phi_i\}$: arbitrary $\psi_1 = \frac{\phi_1}{\|\phi_1\|}$ $\chi_n = \phi_n - \sum_{i=1}^{n-1} \langle \psi_i, \phi_n \rangle \psi_i$ $\psi_n = \frac{\chi_n}{\|\chi_n\|}$

$\{\psi_i\}$: ON $\{\phi_i\} \longrightarrow \{\psi_i\}$: Gram-Schmidt orthogonalisation

$|\langle f, g\rangle| \leq \|f\| \, \|g\|$: Schwarz Inequality

$f = \sum_i a_i \phi_i$; $\|f\|^2 \geq \sum_i |a_i|^2$; Bessel Inequality } If equality, $\{\phi_i\}$: complete

$\sum_i |\phi_i\rangle\langle\phi_i| = I$: closure $A = \sum_i a_i |\phi_i\rangle\langle\phi_i|$ $A^{-1} = \sum_i \frac{1}{a_i} |\phi_i\rangle\langle\phi_i|$

$\langle A^+ f, g\rangle = \langle f, Ag\rangle$ $A = \sum_{i,j} A_{ij} |\phi_i\rangle\langle\phi_j|$; $A_{ij} = \langle\phi_i|A|\phi_j\rangle$

$b = Ac \Rightarrow b_i = \sum_j A_{ij} c_j$ $A^+ = (A^*)^T$, $a^+_{ij} = a^*_{ji}$

$|\phi_i\rangle = \sum_j U_{ji} |\psi_j\rangle$ $U_{ji} = \langle \psi_j, \phi_i \rangle$ $U^{-1} = U^+$ $|\psi_i\rangle = \sum_j U^*_{ij} |\phi_j\rangle$

$\cancel{A : \{|\phi_i\rangle\}, A' : \{|\psi_j\rangle\}}$ $A = \sum_{i,j} A_{ij} |\phi_i\rangle\langle\phi_j|$ $A' = \sum_{i,j} A'_{ij} |\psi_i\rangle\langle\psi_j|$

$A' = UAU^+ = UAU^{-1}$ $\langle f|A|g\rangle$ = constant (under U)

$a = \sum a_i |\phi_i\rangle$ $a' = \sum a_j' |\psi_j\rangle$ $a' = Ua$

$H^+ = H$: hermitian $H|\lambda_i\rangle = \lambda_i |\lambda_i\rangle$ $\lambda_i^* = \lambda_i$: real eigenvalues

If $\{|\lambda_i\rangle\}$ complete, $\{|\lambda_i\rangle\}$: basis (ON) $UHU^{-1} = D$: diagonalisation

$\langle \lambda_i | \lambda_j \rangle = \delta_{ij}$

$D|i\rangle = c_i |i\rangle$ $|i\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow$ i th position c_i : eigenvalues

Commute $\Leftrightarrow$ Compatible (common complete eigenbasis)

$[H_1, H_2] = 0 \Leftrightarrow \{H_1, H_2\}$: CSCO

$[A, A^+] = 0$: A normal $A|v_i\rangle = \lambda_i |v_i\rangle \Rightarrow A^+ |v_i\rangle = \lambda_i^* |v_i\rangle$ $U|v_i\rangle = v_i |v_i\rangle$ $\|v_i\| = 1$