Actuarial Statistics – 2006 Paper

Question 1

1 Consider the total amount of the claims arising from traffic accidents for which an insurance company receives at least one claim. Let N be the number of claims from one such accident and let the claim sizes $X_1, X_2, \ldots$ be independent identically distributed random variables, independent of N. Let $p_n = \mathbb{P}(N = n)$, so that $p_0 = 0$, and assume that

$$p_n = \left(a + \frac{b}{n}\right) p_{n-1}, \quad n = 2, 3, \ldots,$$

where $a, b \in \mathbb{R}$ are known.

Suppose that the claim sizes are discrete with $f_k = \mathbb{P}(X_1 = k)$, $k = 1, 2, \ldots$, where $\sum_{k=1}^{\infty} f_k = 1$, and assume that the p_n's and the f_k's are known. Let $g_k = \mathbb{P}(X_1 + \ldots + X_N = k)$, $k = 1, 2, \ldots$.

By considering probability generating functions, derive a recursion formula for the g_k's in terms of known quantities.

Write down the recursion if

$$p_n = \frac{e^{-\lambda}\lambda^n}{(1 - e^{-\lambda})n!} \quad n = 1, 2, \ldots$$

We begin by noting that since the claim sizes cannot be 0, $g_0 = \mathbb{P}(N = 0) = p_0 = 0$.

We also note that for the compound distribution to have a value of 1, we must have a *single* claim with a value of 1. So $g_1 = f_1 p_1$. This will be the basis for our recursion formula.

Now, mutiply the condition in the question by z^n and sum, to get

$$\sum_{n=1}^{\infty} p_n z^n = \sum_{n=1}^{\infty} z^n \left(a + \frac{b}{n}\right) p_{n-1}$$

$$\sum_{n=0}^{\infty} p_n z^n - p_0 = \sum_{n=1}^{\infty} azz^{n-1} p_{n-1} + b\sum_{n=1}^{\infty} \frac{z^n}{n} p_{n-1}$$

$$G_N(z) - p_0 = az\sum_{n=0}^{\infty} z^n p_n + b\sum_{n=1}^{\infty} \frac{z^n}{n} p_{n-1}$$

$$\left(1 - az\right) G_N(z) = p_0 + b\sum_{n=1}^{\infty} \frac{z^n}{n} p_{n-1}$$

Differentiating with respect to z

$$-aG_N(z) + \left(1 - az\right) G_N'(z) = bG_N(z)$$

$$G_N'(z) = \frac{a + b}{1 - az} G_N(z)$$

Now, let

The total number of claims is given by

$$N = N_1 + \cdots + N_n$$

Consider the MGF of N:

$$M_N(t) = \mathbb{E}\left(e^{Nt}\right) = \mathbb{E}\left(e^{N_1 t}\right) \cdots \mathbb{E}\left(e^{N_n t}\right) = M_{N_1}(t) \cdots M_{N_n}(t)$$

Furthermore, since N_i has a negative binomial distribution,

$$M_{N_i}\left(t\right) = \left(\frac{1}{1 - \frac{1}{\alpha+1} e^t}\right)^m$$

And so

$$M_N\left(t\right) = \left(\frac{1}{1 - \frac{1}{\alpha+1} e^t}\right)^{mn}$$

This is also the MGF of a negative binomial, with the same p parameter but with

$$p = \tfrac{\alpha}{\alpha+1}$$
$$r = nm$$

In other words

$$N \sim \text{NegBin}\left(r = nm, p = \tfrac{\alpha}{\alpha+1}\right)$$

This can be viewed as a single Poisson distribution $N \sim \text{Po}(\lambda)$ with its parameter mixed over the following distribution:

$$f(\lambda) = \frac{\alpha^{nm} \lambda^{nm-1} e^{-\alpha\lambda}}{(nm-1)!} \qquad \lambda > 0 \qquad (*)$$

Now, the claims sizes *all* have exponential distribution with parameter μ. And we have just seen that the total number of claims in a year is N, across all categories. Now

$$S = \sum_{k=1}^{n} \sum_{i=1}^{N_k} X_i = \sum_{i=1}^{N} X_i$$

So S does indeed have a compound mixed Poisson distribution, and the mixing distribution is that in equation (*).