$4 \cdot 0 \equiv 0 \pmod 5$. Thus the number of possible sequences $a_1, a_2, \dots$ is 0 or 2 (mod 5), as desired.

**Second solution.** Say that a sequence is *admissible* if it satisfies the given conditions. As in the first solution, any admissible sequence is 5-periodic.

Now consider the collection $S$ of possible 5-tuples of numbers mod $p$ given by $(a_1,a_2,a_3,a_4,a_5)$ for admissible sequences $\{a_n\}$. Each of these 5-tuples in $S$ comes from a unique admissible sequence, and there is a 5-periodic action on $S$ given by cyclic permutation: $(a,b,c,d,e) \to (b,c,d,e,a)$. This action divides $S$ into finitely many orbits, and each orbit either consists of 5 distinct tuples (if $a,b,c,d,e$ are not all the same) or 1 tuple $(a,a,a,a,a)$. It follows that the number of admissible sequences is a multiple of 5 plus the number of constant admissible sequences.

Constant admissible sequences correspond to nonzero numbers $a \pmod p$ such that $a^2 \equiv 1+a \pmod p$. Since the quadratic $x^2-x-1$ has discriminant 5, for $p>5$ it has either 2 roots (if the discriminant is a quadratic residue mod $p$) or 0 roots mod $p$.

A4 The expected value is $2e^{1/2}-3$.

Extend $S$ to an infinite sum by including zero summands for $i>k$. We may then compute the expected value as the sum of the expected value of the $i$-th summand over all $i$. This summand occurs if and only if $X_1,\dots,X_{i-1} \in [X_i,1]$ and $X_1,\dots,X_{i-1}$ occur in nonincreasing order. These two events are independent and occur with respective probabilities $(1-X_i)^{i-1}$ and $\frac{1}{(i-1)!}$; the expectation of this summand is therefore

$$\begin{aligned}
&\frac{1}{2^i(i-1)!}\int_0^1 t(1-t)^{i-1}\,dt \\
&\quad = \frac{1}{2^i(i-1)!}\int_0^1 ((1-t)^{i-1}-(1-t)^i)\,dt \\
&\quad = \frac{1}{2^i(i-1)!}\left(\frac{1}{i}-\frac{1}{i+1}\right) = \frac{1}{2^i(i+1)!}.
\end{aligned}$$

Summing over $i$, we obtain

$$\sum_{i=1}^{\infty}\frac{1}{2^i(i+1)!} = 2\sum_{i=2}^{\infty}\frac{1}{2^i i!} = 2\left(e^{1/2}-1-\frac{1}{2}\right).$$

A5 We show that the number in question equals 290. More generally, let $a(n)$ (resp. $b(n)$) be the optimal final score for Alice (resp. Bob) moving first in a position with $n$ consecutive squares. We show that

$$\begin{aligned}
a(n) &= \left\lfloor \frac{n}{7} \right\rfloor + a\left(n-7\left\lfloor \frac{n}{7}\right\rfloor\right), \\
b(n) &= \left\lfloor \frac{n}{7} \right\rfloor + b\left(n-7\left\lfloor \frac{n}{7}\right\rfloor\right),
\end{aligned}$$

and that the values for $n \leq 6$ are as follows:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $a(n)$ | 0 | 1 | 0 | 1 | 2 | 1 | 2 |
| $b(n)$ | 0 | 1 | 0 | 1 | 0 | 1 | 0 |

Since $2022 \equiv 6 \pmod 7$, this will yield $a(2022) = 2 + \lfloor \frac{2022}{7} \rfloor = 290$.

We proceed by induction, starting with the base cases $n \leq 6$. Since the number of odd intervals never decreases, we have $a(n), b(n) \geq n - 2\lfloor \frac{n}{2} \rfloor$; by looking at the possible final positions, we see that equality holds for $n = 0,1,2,3,5$. For $n=4,6$, Alice moving first can split the original interval into two odd intervals, guaranteeing at least two odd intervals in the final position; whereas Bob can move to leave behind one or two intervals of length 2, guaranteeing no odd intervals in the final position.

We now proceed to the induction step. Suppose that $n \geq 7$ and the claim is known for all $m<n$. In particular, this means that $a(m) \geq b(m)$; consequently, it does not change the analysis to allow a player to pass their turn after the first move, as both players will still have an optimal strategy which involves never passing.

It will suffice to check that

$$a(n) = a(n-7)+1, \qquad b(n) = b(n-7)+1.$$

Moving first, Alice can leave behind two intervals of length 1 and $n-3$. This shows that

$$a(n) \geq 1 + b(n-3) = a(n-7)+1.$$

On the other hand, if Alice leaves behind intervals of length $i$ and $n-2-i$, Bob can choose to play in either one of these intervals and then follow Alice's lead thereafter (exercising the pass option if Alice makes the last legal move in one of the intervals). This shows that

$$\begin{aligned}
a(n) &\leq \max\{\min\{a(i)+b(n-2-i), \\
&\qquad b(i)+a(n-2-i)\} : i = 0,1,\dots,n-2\} \\
&= a(n-7)+1.
\end{aligned}$$

Moving first, Bob can leave behind two intervals of lengths 2 and $n-4$. This shows that

$$b(n) \leq a(n-4) = b(n-7)+1.$$

On the other hand, if Bob leaves behind intervals of length $i$ and $n-2-i$, Alice can choose to play in either one of these intervals and then follow Bob's lead thereafter (again passing as needed). This shows that

$$\begin{aligned}
b(n) &\geq \min\{\max\{a(i)+b(n-2-i), \\
&\qquad b(i)+a(n-2-i)\} : i = 0,1,\dots,n-2\} \\
&= b(n-7)+1.
\end{aligned}$$

This completes the induction.

A6 **First solution.** The largest such $m$ is $n$. To show that $m \geq n$, we take

$$x_j = \cos\frac{(2n+1-j)\pi}{2n+1} \qquad (j = 1,\dots,2n).$$