B2 Determine the maximum value of the sum

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} \left(a_1 a_2 \cdots a_n\right)^{1/n}$$

over all sequences $a_1, a_2, a_3, \cdots$ of nonnegative real numbers satisfying

$$\sum_{k=1}^{\infty} a_k = 1 \, .$$

Answer: The maximum value is $S = \dfrac{2}{3}$; it is achieved by the sequence $a_k = \dfrac{3}{4^k}$.

Solution: First consider geometric sequences, which are given by $a_k = a_1 r^{k-1}$ for all k, with $0 < r < 1$. For such a sequence we have

$$\left(a_1 a_2 \cdots a_n\right)^{1/n} = a_1 \left(1 \cdot r \cdot \, \cdots \, \cdot r^{n-1}\right)^{1/n} = a_1 \left(r^{n(n-1)/2}\right)^{1/n} = a_1 \, r^{(n-1)/2} \, ,$$

and the constraint $\sum\limits_{k=1}^{\infty} a_k = 1$ yields $a_1 = 1 - r$. Thus we can calculate S as a function of r:

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{n}{2^n} \left(a_1 a_2 \cdots a_n\right)^{1/n} = (1-r) \sum_{n=1}^{\infty} \frac{n r^{(n-1)/2}}{2^n} = \frac{1-r}{\sqrt{r}} \sum_{n=1}^{\infty} n \left(\frac{\sqrt{r}}{2}\right)^n \\ &= \frac{1-r}{\sqrt{r}} f\left(\frac{\sqrt{r}}{2}\right) = \frac{2(1-r)}{(2-\sqrt{r})^2} \, , \quad \text{where } \; f(x) = \sum_{n=1}^{\infty} n x^n = x \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2} \, . \end{aligned}$$

By taking the derivative of S with respect to r, which is zero only for $r = \dfrac{1}{4}$, and comparing the values of $\dfrac{2(1-r)}{(2-\sqrt{r})^2}$ for $r = 0$, $r = \dfrac{1}{4}$, and $r = 1$, we find that the maximum value of S that can be obtained for a geometric sequence is $\dfrac{2(3/4)}{(3/2)^2} = \dfrac{2}{3}$, for $r = \dfrac{1}{4}$. It remains to show that this is actually the maximum value for *any* sequence.

Given any sequence of nonnegative numbers that sum to 1, consider the geometric mean, say G_n , of the first n numbers. This can be written as

$$G_n = \left(a_1 a_2 \cdots a_n\right)^{1/n} = \left[\frac{(4a_1) \cdot (4^2 a_2) \, \cdots \, (4^n a_n)}{4^1 \cdot 4^2 \, \cdots \, 4^n}\right]^{1/n} = \frac{1}{2^{n+1}} \left[(4a_1) \cdot (4^2 a_2) \, \cdots \, (4^n a_n)\right]^{1/n} ,$$

and we can then apply the AM-GM inequality to obtain

$$G_n \le \frac{1}{2^{n+1}} \frac{\left[(4a_1) + (4^2 a_2) + \, \cdots \, + (4^n a_n)\right]}{n} = \frac{1}{n 2^{n+1}} \sum_{k=1}^{n} 4^k a_k \, .$$

We then have

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} G_n \le \sum_{n=1}^{\infty} \left(\frac{n}{2^n} \cdot \frac{1}{n 2^{n+1}} \sum_{k=1}^{n} 4^k a_k\right) = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{a_k}{4^{n-k}} \, .$$

This series is absolutely convergent, so we can change the order of summation to get

$$S \le \frac{1}{2} \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{a_k}{4^{n-k}} = \frac{1}{2} \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{a_k}{4^j} = \frac{1}{2} \left[\sum_{j=0}^{\infty} \frac{1}{4^j}\right] \left[\sum_{k=1}^{\infty} a_k\right] .$$

The first bracketed factor is a geometric series with sum $\dfrac{1}{1-\frac{1}{4}} = \dfrac{4}{3}$ and the second factor is 1 by the given constraint, so $S \le \frac{2}{3}$, and we are done.

B5 Say that an n-by-n matrix $A = (a_{ij})_{1 \le i,j \le n}$ with integer entries is *very odd* if, for every nonempty subset S of $\{1, 2, \ldots, n\}$, the $|S|$-by-$|S|$ submatrix $(a_{ij})_{i,j \in S}$ has odd determinant. Prove that if A is very odd, then A^k is very odd for every $k \ge 1$.

Solution: First of all, because we are only interested in determinants modulo 2, we can reduce the entries of A modulo 2; that is, we may assume that all entries of A are in $\{0, 1\}$.

Claim: Under this assumption, a necessary and sufficient condition for A to be very odd is that there exists a permutation π of $\{1, \ldots, n\}$ such that, when both the rows and columns of A are permuted by π, A becomes upper triangular with all diagonal entries 1. In other words, A is very odd if and only if there exists an n-by-n permutation matrix P such that PAP^{-1} is upper triangular with 1's along the diagonal.

Note that if PAP^{-1} is upper triangular with 1's along the diagonal, then so is $PA^kP^{-1} = (PAP^{-1})^k$. Therefore, the problem statement follows immediately from the claim.

Proof of the claim: To show the condition is sufficient, note that if A is upper triangular with 1's on the diagonal, then any submatrix $(a_{ij})_{i,j \in S}$ has that same form, so such a submatrix has determinant 1. Also, permuting the rows and columns of A by a permutation π does not affect the set of determinants of the submatrices.

Now we show the condition is necessary. Suppose that A is very odd (and has entries from $\{0, 1\}$). By taking the subsets $S = \{i\}$ of $\{1, \ldots, n\}$, we see that $a_{ii} = 1$ for all i. Now consider a two-element subset $\{i, j\}$. Because the determinant $a_{ii}a_{jj} - a_{ij}a_{ji}$ must be odd, at least one of a_{ij} and a_{ji} must be zero. Define a relation $\lhd$ on $\{1, \ldots, n\}$ by

$$i \lhd j \quad \text{if and only if} \quad a_{ij} = 1.$$

Then we've seen that for $i \ne j$, we cannot have both $i \lhd j$ and $j \lhd i$. In fact, we'll show that the relation $\lhd$ is *acyclic*, meaning that there is no cycle $i_1 \lhd i_2 \lhd \cdots \lhd i_k \lhd i_1$ with $k > 1$ (and $i_1 \ne i_2$). Suppose we do have such a cycle, and take one for which k is as small as possible. Consider the submatrix $M = (a_{ij})_{i,j \in S}$ of A corresponding to the subset $S = \{i_1, i_2, \ldots, i_k\}$. Then in the expression of $\det(M)$ as a sum of signed products of entries of M, each corresponding to a permutation of S, there will be exactly two nonzero terms, namely the "diagonal" term $a_{i_1 i_1} a_{i_2 i_2} \cdots a_{i_k i_k} = 1$ and a term $\pm a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1} = \pm 1$ corresponding to the cycle. (Any nonzero term in the determinant has to be $\pm$ a product of 1's, and unless the corresponding permutation is the identity it has at least one nontrivial cycle in its cycle decomposition, which is then a cycle for $\lhd$; because k is as small as possible, this can only be a k-cycle, which means it must involve all the elements of S, and if it weren't the original cycle $(i_1\, i_2 \cdots i_k)$, it could be used together with the original cycle to construct a shorter cycle for $\lhd$.) But then $\det(M)$ is even, which is a contradiction.

Because $\lhd$ is acyclic, we can find a permutation π of $\{1, \ldots, n\}$ such that $i \lhd j$ implies $\pi(i) \le \pi(j)$. If we then use π to rearrange the rows and columns of A, the new matrix will have the desired upper triangular form with 1's on the diagonal. (An explicit procedure for constructing π is as follows: List the elements of $\{1, \ldots, n\}$ in stages, starting with the elements - in any order - that have no "predecessors" under the relation $\lhd$. At each subsequent stage, list, in any order, the elements all of whose predecessors have already been listed. When the list is complete, let $\pi(i)$ be the ith number on the list.)

B6 Given an ordered list of $3N$ real numbers, we can *trim* it to form a list of N numbers as follows: We divide the list into N groups of 3 consecutive numbers, and within each group, discard the highest and lowest numbers, keeping only the median.

Consider generating a random number X by the following procedure: Start with a list of 3^{2021} numbers, drawn independently and uniformly at random between 0 and 1. Then trim this list as defined above, leaving a list of 3^{2020} numbers. Then trim again repeatedly until just one number remains; let X be this number. Let μ be the expected value of $|X - \frac{1}{2}|$. Show that

$$\mu \geq \frac{1}{4}\left(\frac{2}{3}\right)^{2021}.$$

Solution: First, replace each random number x by $z = x - 1/2$, which will lie in the interval $[-1/2, 1/2]$. Let $\rho_n(z)$ be the probability density function on that interval for each of the numbers that remain after n trims. We know that $\rho_0(z) = 1$ because the initial distribution is uniform. Furthermore, $\rho_n(-z) = \rho_n(z)$ for all n, as the process is now symmetric with respect to the origin. This implies that

$$\int_{-\frac{1}{2}}^{0} \rho_n(t)\,dt = \int_{0}^{\frac{1}{2}} \rho_n(t)\,dt = \frac{1}{2}\,.$$

We proceed to calculate ρ_n , the probability density after n trims, from ρ_{n-1}. When we carry out the nth trim, there are $3! = 6$ equivalent orderings of the three numbers in a group, so we may first assume a fixed ordering of these numbers (specifically, let the first be the median, the second be the smallest, and the third be the largest) and then multiply by 6 to take the possible orderings into account. This yields the recursive formula

$$\begin{aligned}
\rho_n(z) &= 6\,\rho_{n-1}(z)\left[\int_{-\frac{1}{2}}^{z} \rho_{n-1}(t)\,dt\right]\left[\int_{z}^{\frac{1}{2}} \rho_{n-1}(t)\,dt\right] \\
&= 6\,\rho_{n-1}(z)\left[\frac{1}{2} + \int_{0}^{z} \rho_{n-1}(t)\,dt\right]\left[\frac{1}{2} - \int_{0}^{z} \rho_{n-1}(t)\,dt\right] \\
&= \frac{3}{2}\,\rho_{n-1}(z)\left[1 - 4\left(\int_{0}^{z} \rho_{n-1}(t)\,dt\right)^2\right].
\end{aligned}$$

It follows that $\rho_n(0) = \frac{3}{2}\rho_{n-1}(0)$, so by induction on n we have $\rho_n(0) = \left(\frac{3}{2}\right)^n$. Also by induction, for $n \geq 1$ the function $\rho_n(z)$ is monotonically decreasing with respect to $|z|$, and in particular $\rho_n(z) \leq \rho_n(0) = \left(\frac{3}{2}\right)^n$.

Now let $n = 2021$, so the expected value μ in the problem is given by

$$\mu = \int_{-1/2}^{1/2} |z|\rho_n(z)\,dz = 2\int_{0}^{1/2} z\rho_n(z)\,dz\,.$$

Let $M = \rho_n(0) = \left(\frac{3}{2}\right)^{2021}$. For $0 \leq z \leq \frac{1}{2}$, define the antiderivative

$$S(z) = \int_{t=0}^{z} \rho_n(t)\,dt \quad \text{of} \quad \rho_n(z)\,;$$

note that

$$S(0) = 0,\; S(\tfrac{1}{2}) = \int_{0}^{\frac{1}{2}} \rho_n(t)\,dt = \frac{1}{2}\,,$$