B5 **Solution 1.** We prove that $(j,k) = (2019,1010)$ is a valid solution. More generally, let $p(x)$ be the polynomial of degree N such that $p(2n+1) = F_{2n+1}$ for $0 \leq n \leq N$. We will show that $p(2N+3) = F_{2N+3} - F_{N+2}$.

Define a sequence of polynomials $p_0(x),\dots,p_N(x)$ by $p_0(x) = p(x)$ and $p_k(x) = p_{k-1}(x) - p_{k-1}(x+2)$ for $k \geq 1$. Then by induction on k, it is the case that $p_k(2n+1) = F_{2n+1+k}$ for $0 \leq n \leq N-k$, and also that p_k has degree (at most) $N-k$ for $k \geq 1$. Thus $p_N(x) = F_{N+1}$ since $p_N(1) = F_{N+1}$ and p_N is constant.

We now claim that for $0 \leq k \leq N$, $p_{N-k}(2k+3) = \sum_{j=0}^{k} F_{N+1+j}$. We prove this again by induction on k: for the induction step, we have

$$\begin{aligned} p_{N-k}(2k+3) &= p_{N-k}(2k+1) + p_{N-k+1}(2k+1) \\ &= F_{N+1+k} + \sum_{j=0}^{k-1} F_{N+1+j}. \end{aligned}$$

Thus we have $p(2N+3) = p_0(2N+3) = \sum_{j=0}^{N} F_{N+1+j}$.

Now one final induction shows that $\sum_{j=1}^{m} F_j = F_{m+2} - 1$, and so $p(2N+3) = F_{2N+3} - F_{N+2}$, as claimed. In the case $N = 1008$, we thus have $p(2019) = F_{2019} - F_{1010}$.

Solution 2. This solution uses the *Lagrange interpolation formula*: given $x_0,\dots,x_n$ and $y_0,\dots,y_n$, the unique polynomial P of degree at most n satisfying $P(x_i) = y_i$ for $i = 0,\dots,n$ is

$$\sum_{i=0}^{n} P(x_i) \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} =$$

Write

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^{-n}), \qquad \alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}.$$

For $\gamma \in \mathbb{R}$, let $p_\gamma(x)$ be the unique polynomial of degree at most 1008 satisfying

$$p_1(2n+1) = \gamma^{2n+1}, p_2(2n+1) = \gamma^{2n+1}\ (n = 0,\dots,1008);$$

then $p(x) = \frac{1}{\sqrt{5}}(p_\alpha(x) - p_\beta(x))$.

By Lagrange interpolation,

$$\begin{aligned} p_\gamma(2019) &= \sum_{n=0}^{1008} \gamma^{2n+1} \prod_{0 \leq j \leq 1008, j \neq n} \frac{2019 - (2j+1)}{(2n+1) - (2j+1)} \\ &= \sum_{n=0}^{1008} \gamma^{2n+1} \prod_{0 \leq j \leq 1008, j \neq n} \frac{1009 - j}{n - j} \\ &= \sum_{n=0}^{1008} \gamma^{2n+1} (-1)^{1008-n} \binom{1009}{n} \\ &= -\gamma((\gamma^2 - 1)^{1009} - (\gamma^2)^{1009}). \end{aligned}$$

For $\gamma \in \{\alpha, \beta\}$ we have $\gamma^2 = \gamma + 1$ and so

$$p_\gamma(2019) = \gamma^{2019} - \gamma^{1010}.$$

We thus deduce that $p(x) = F_{2019} - F_{1010}$ as claimed.

Remark. Karl Mahlburg suggests the following variant of this. As above, use Lagrange interpolation to write

$$p(2019) = \sum_{j=0}^{1008} \binom{1009}{j} F_j;$$

it will thus suffice to verify (by substiting $j \mapsto 1009 - j$) that

$$\sum_{j=0}^{1009} \binom{1009}{j} F_{j+1} = F_{2019}.$$

This identity has the following combinatorial interpretation. Recall that F_{n+1} counts the number of ways to tile a $1 \times n$ rectangle with 1×1 squares and 1×2 dominoes (see below). In any such tiling with $n = 2018$, let j be the number of squares among the first 1009 tiles. These can be ordered in $\binom{1009}{j}$ ways, and the remaining $2018 - j - 2(1009 - j) = j$ squares can be tiled in F_{j+1} ways.

As an aside, this interpretation of F_{n+1} is the oldest known interpretation of the Fibonacci sequence, long predating Fibonacci himself. In ancient Sanskrit, syllables were classified as long or short, and a long syllable was considered to be twice as long as a short syllable; consequently, the number of syllable patterns of total length n equals F_{n+1}.

Remark. It is not difficult to show that the solution $(j,k) = (2019,2010)$ is unique (in positive integers). First, note that to have $F_j - F_k > 0$, we must have $k < j$. If $j < 2019$, then

$$F_{2019} - F_{1010} = F_{2018} + F_{2017} - F_{1010} > F_j > F_j - F_k.$$

If $j > 2020$, then

$$F_j - F_k \geq F_j - F_{j-1} = F_{j-2} \geq F_{2019} > F_{2019} - F_{1010}.$$

Since $j = 2019$ obviously forces $k = 1010$, the only other possible solution would be with $j = 2020$. But then

$$(F_j - F_k) - (F_{2019} - F_{1010}) = (F_{2018} - F_k) + F_{1010}$$

which is negative for $k = 2019$ (it equals $F_{1010} - F_{2017}$) and positive for $k \leq 2018$.

B6 Such a set exists for every n. To construct an example, define the function $f : \mathbb{Z}^n \to \mathbb{Z}/(2n+1)\mathbb{Z}$ by

$$f(x_1,\dots,x_n) = x_1 + 2x_2 + \cdots + nx_n \pmod{2n+1},$$

then let S be the preimage of 0.

To check condition (1), note that if $p \in S$ and q is a neighbor of p differing only in coordinate i, then

$$f(q) = f(p) \pm i \equiv \pm i \pmod{2n+1}$$

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