B1. Let $\mathcal{P}$ be the set of vectors defined by

$$\mathcal{P} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \,\middle|\, 0 \le a \le 2,\ 0 \le b \le 100,\ \text{and } a, b \in \mathbb{Z} \right\}.$$

Find all $\mathbf{v} \in \mathcal{P}$ such that the set $\mathcal{P} \setminus \{\mathbf{v}\}$ obtained by omitting vector $\mathbf{v}$ from $\mathcal{P}$ can be partitioned into two sets of equal size and equal sum.

Answer. The vectors $\mathbf{v}$ of the form $\begin{pmatrix} 1 \\ b \end{pmatrix}$ with b even, $0 \le b \le 100$.

Solution. First note that if we add all the vectors in $\mathcal{P}$ by first summing over a for fixed b, we get the sum of $\begin{pmatrix} 3 \\ 3b \end{pmatrix}$ for $0 \le b \le 100$, which is $\begin{pmatrix} 303 \\ 3 \cdot (1 + \cdots + 100) \end{pmatrix} = \begin{pmatrix} 303 \\ 3 \cdot 50 \cdot 101 \end{pmatrix}$. Thus if the set $\mathcal{P} \setminus \{\mathbf{v}\}$ is to be partitioned into two sets of equal sum, the vector $\begin{pmatrix} 303 \\ 3 \cdot 50 \cdot 101 \end{pmatrix} - \mathbf{v}$ must have both coordinates even. For $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, this implies that a is odd and b is even, so because $\mathbf{v} \in \mathcal{P}$, we have $a = 1, b$ even, $0 \le b \le 100$. It remains to show that this necessary condition on $\mathbf{v}$ is also sufficient.

Identify each of the vectors $\mathbf{w} = \begin{pmatrix} c \\ d \end{pmatrix}$ in $\mathcal{P}$ with the lattice point (c, d). Given a particular vector $\mathbf{v} = \begin{pmatrix} 1 \\ b \end{pmatrix}$ in $\mathcal{P}$ with b even, there are 302 lattice points in $\mathcal{P} \setminus \{\mathbf{v}\}$. If we can number these points $P_1, P_2, \ldots, P_{302}$ such that the sum of the displacement vectors $\overrightarrow{P_1P_2}, \overrightarrow{P_3P_4}, \ldots, \overrightarrow{P_{301}P_{302}}$ is zero, then we can partition $\mathcal{P} \setminus \{\mathbf{v}\}$ into the set of points P_i with i odd and the set of points P_i with i even, and those sets will have equal size and equal sum. To do so we first partition the set $\mathcal{P} \setminus \{\mathbf{v}\}$ into 24 rectangular sets of 4×3 lattice points, along with a single 5×3 rectangular set from which one of the points in the middle column is missing. For the 4×3 rectangular sets, we can take three displacement vectors pointing up and three displacement vectors pointing down, as shown in the first diagram below. For the 5×3 rectangular set with a single point missing, there are (up to symmetry) two cases, depending on whether the missing point is on an edge or at the center (the parity condition on b guarantees that it will be either on an edge or at the center). These are shown in the second and third diagrams below. As an example, if the 5×3 rectangle is the set

$$\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \,\middle|\, 0 \le a \le 2,\ 0 \le b \le 4,\ \text{and } a, b \in \mathbb{Z} \right\}$$

and $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the missing point, then the second diagram corresponds to the partition of the rectangular set minus that point into the two subsets of equal size and equal sum

$$\{(0,0), (0,2), (0,4), (1,2), (2,4), (2,2), (2,1)\} \quad \text{and}$$
$$\{(0,1), (0,3), (1,4), (1,1), (2,3), (1,3), (2,0)\},$$

which contain the starting and end points, respectively, of the displacement vectors shown.

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