with $\delta > 0$ as small as possible. If a term in S_1 is exactly one less than a term in S_2, then we can exchange those terms and make δ smaller, contradiction. So because all scores occur, if the smallest term in S_1 is a, then $a+1$ must also appear in S_1, and repeating this argument, $a+1, a+2, \ldots$ must all appear in S_1. Let the largest term in S_2 be b. Then $5ma \leq S_1 < S_2 \leq 5mb$, so $b > a$, so $b-1$ must appear in S_1 and we can exchange $b-1$ from S_1 and b from S_2 after all to reduce δ, contradiction.

A5. Each of the integers from 1 to n is written on a separate card, and then the cards are combined into a deck and shuffled. Three players, A, B, and C, take turns in the order $A, B, C, A, \ldots$ choosing one card at random from the deck. (Each card in the deck is equally likely to be chosen.) After a card is chosen, that card and all higher-numbered cards are removed from the deck, and the remaining cards are reshuffled before the next turn. Play continues until one of the three players wins the game by drawing the card numbered 1.

Show that for each of the three players, there are arbitrarily large values of n for which that player has the highest probability among the three players of winning the game.

Solution. For every positive integer n, let A_n, B_n, C_n denote the probabilities that players A, B, C (respectively) win the game for that value of n. Note that $A_1 = 1$, $B_1 = C_1 = 0$ (if $n = 1$, there is only one card, and A gets to choose it). For $n > 1$, if player A chooses the card numbered k with $k > 1$, the game then proceeds like the original game with $k-1$ cards, except that B now chooses first, C chooses second, and A takes on C's original role, choosing third. Therefore, we have the recurrence relations

$$\begin{aligned}
A_n &= \frac{1}{n} + \frac{1}{n}\,C_1 + \frac{1}{n}\,C_2 + \cdots + \frac{1}{n}\,C_{n-1}\,, \\
B_n &= \frac{1}{n}\,A_1 + \frac{1}{n}\,A_2 + \cdots + \frac{1}{n}\,A_{n-1}\,, \\
C_n &= \frac{1}{n}\,B_1 + \frac{1}{n}\,B_2 + \cdots + \frac{1}{n}\,B_{n-1}\,.
\end{aligned}$$

Multiplying through by n and then subtracting the equations for n from those for $n+1$ yields

$$(n+1)\,A_{n+1} - n\,A_n = C_n\,, \quad (n+1)\,B_{n+1} - n\,B_n = A_n\,, \quad (n+1)\,C_{n+1} - n\,C_n = B_n\,.$$

Thus we have

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \\ C_{n+1} \end{pmatrix} = M_n \begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix}, \quad \text{where} \quad M_n = \frac{1}{n+1} \begin{pmatrix} n & 0 & 1 \\ 1 & n & 0 \\ 0 & 1 & n \end{pmatrix}.$$

The eigenvalues of the matrix $\begin{pmatrix} n & 0 & 1 \\ 1 & n & 0 \\ 0 & 1 & n \end{pmatrix}$ are the roots of $(n-\lambda)^3 + 1 = 0$, so if we let $\omega = e^{2\pi i/3}$, they are given by $n - \lambda = -1$, $n - \lambda = -\omega$, $n - \lambda = -\omega^2$. Dividing by

$n+1$, we find the eigenvalues

$$1,\ \frac{n+\omega}{n+1},\ \frac{n+\omega^2}{n+1}$$

of M_n, and a straightforward computation yields corresponding eigenvectors

$$\begin{pmatrix}1\\1\\1\end{pmatrix},\ \begin{pmatrix}1\\\omega^2\\\omega\end{pmatrix},\ \begin{pmatrix}1\\\omega\\\omega^2\end{pmatrix}$$

respectively. In particular, the eigenvectors are the same for each n, and so we can use them, together with

$$\begin{pmatrix}A_{n+1}\\B_{n+1}\\C_{n+1}\end{pmatrix} = M_n \begin{pmatrix}A_n\\B_n\\C_n\end{pmatrix},$$

to find expressions for the probabilities A_n, B_n, C_n, as follows:

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\\omega^2\\\omega\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\\omega\\\omega^2\end{pmatrix}, \quad \text{so}$$

$$\begin{aligned}
\begin{pmatrix}A_n\\B_n\\C_n\end{pmatrix} &= M_{n-1}M_{n-2}\cdots M_1\begin{pmatrix}1\\0\\0\end{pmatrix}\\
&= \frac{1}{3}M_{n-1}M_{n-2}\cdots M_1\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{3}M_{n-1}M_{n-2}\cdots M_1\begin{pmatrix}1\\\omega^2\\\omega\end{pmatrix}\\
&\qquad + \frac{1}{3}M_{n-1}M_{n-2}\cdots M_1\begin{pmatrix}1\\\omega\\\omega^2\end{pmatrix}\\
&= \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{3}\prod_{k=1}^{n-1}\frac{k+\omega}{k+1}\cdot\begin{pmatrix}1\\\omega^2\\\omega\end{pmatrix} + \frac{1}{3}\prod_{k=1}^{n-1}\frac{k+\omega^2}{k+1}\cdot\begin{pmatrix}1\\\omega\\\omega^2\end{pmatrix}.
\end{aligned}$$

Let $P = \prod_{k=1}^{n-1}\frac{k+\omega}{k+1}$. Then, because $\omega^2 = \overline{\omega}$, we have

$$\begin{aligned}
A_n &= \frac{1}{3}(1+P+\overline{P}) = \frac{1}{3}(1+2\,\mathrm{Re}(P)),\\
B_n &= \frac{1}{3}(1+P\overline{\omega}+\overline{P}\omega) = \frac{1}{3}(1+2\,\mathrm{Re}(P\overline{\omega})),\\
C_n &= \frac{1}{3}(1+P\omega+\overline{P\omega}) = \frac{1}{3}(1+2\,\mathrm{Re}(P\omega)).
\end{aligned}$$

Thus, which of the three players has the highest probability of winning the game depends only on which of $\mathrm{Re}(P)$, $\mathrm{Re}(P\overline{\omega})$, $\mathrm{Re}(P\omega)$ is the largest; that, in turn, depends only on the argument of P. Specifically, A_n is largest when $\mathrm{Arg}(P)$ is in the interval $[-\pi/3, \pi/3]$, B_n is largest when $\mathrm{Arg}(P)$ is in $[\pi/3, \pi]$, and C_n is largest when $\mathrm{Arg}(P)$