corresponding to the cases where $\pi(1), \pi(2) = 1, 2$; where $\pi(1), \pi(2), \pi(3) = 1, 3, 2$; and the unique case $1, 3, 5, \dots, 6, 4, 2$. Meanwhile, one has

$$R'_n = R'_{n-1} + Q'_{n-2}$$

corresponding to the cases containing $3, 1, 2, 4$ (where removing 1 and reversing gives a permutation counted by R'_{n-1}); and where 4 occurs before $3, 1, 2$ (where removing $1, 2$ and reversing gives a permutation counted by Q'_{n-2}).

Remark: The permutations counted by P_n are known as *key permutations*, and have been studied by E.S. Page, Systematic generation of ordered sequences using recurrence relations, *The Computer Journal* **14** (1971), no. 2, 150–153. We have used the same notation for consistency with the literature. The sequence of the P_n also appears as entry A003274 in the On-line Encyclopedia of Integer Sequences (`http://oeis.org`).

B6 (from `artofproblemsolving.com`) We will prove that the sum converges to $\pi^2/16$. Note first that the sum does not converge absolutely, so we are not free to rearrange it arbitrarily. For that matter, the standard alternating sum test does not apply because the absolute values of the terms does not decrease to 0, so even the convergence of the sum must be established by hand.

Setting these issues aside momentarily, note that the elements of the set counted by $A(k)$ are those odd positive integers d for which $m = k/d$ is also an integer and $d < \sqrt{2dm}$; if we write $d = 2\ell - 1$, then the condition on m reduces to $m \geq \ell$. In other words, the original sum equals

$$S_1 := \sum_{k=1}^\infty \sum_{\substack{\ell \geq 1, m \geq \ell \\ k = m(2\ell-1)}} \frac{(-1)^{m-1}}{m(2\ell-1)},$$

and we would like to rearrange this to

$$S_2 := \sum_{\ell=1}^\infty \frac{1}{2\ell-1} \sum_{m=\ell}^\infty \frac{(-1)^{m-1}}{m},$$

in which both sums converge by the alternating sum test. In fact a bit more is true: we have

$$\left| \sum_{m=\ell}^\infty \frac{(-1)^{m-1}}{m} \right| < \frac{1}{\ell},$$

so the outer sum converges absolutely. In particular, S_2 is the limit of the truncated sums

$$S_{2,n} = \sum_{\ell(2\ell-1) \leq n} \frac{1}{2\ell-1} \sum_{m=\ell}^\infty \frac{(-1)^{m-1}}{m}.$$

To see that S_1 converges to the same value as S_2, write

$$S_{2,n} - \sum_{k=1}^n (-1)^{k-1} \frac{A(k)}{k} = \sum_{\ell(2\ell-1) \leq n} \frac{1}{2\ell-1} \sum_{m=\lfloor \frac{n}{2\ell-1}+1 \rfloor}^\infty \frac{(-1)^{m-1}}{m}.$$

The expression on the right is bounded above in absolute value by the sum $\sum_{\ell(2\ell-1) \leq n} \frac{1}{n}$, in which the number of summands is at most $\sqrt{n}$ (since $\sqrt{n}(2\sqrt{n}-1) \geq n$), and so the total is bounded above by $1/\sqrt{n}$. Hence the difference converges to zero as $n \to \infty$; that is, S_1 converges and equals S_2.

We may thus focus hereafter on computing S_2. We begin by writing

$$S_2 = \sum_{\ell=1}^\infty \frac{1}{2\ell-1} \sum_{m=\ell}^\infty (-1)^{m-1} \int_0^1 t^{m-1}\,dt.$$

Our next step will be to interchange the inner sum and the integral, but again this requires some justification.

Lemma 1. *Let $f_0, f_1, \dots$ be a sequence of continuous functions on $[0,1]$ such that for each $x \in [0,1]$, we have*

$$f_0(x) \geq f_1(x) \geq \cdots \geq 0.$$

Then

$$\sum_{n=0}^\infty (-1)^n \int_0^1 f_n(t)\,dt = \int_0^1 \left(\sum_{n=0}^\infty (-1)^n f_n(t) \right) dt$$

provided that both sums converge.

Proof. Put $g_n(t) = f_{2n}(t) - f_{2n+1}(t) \geq 0$; we may then rewrite the desired equality as

$$\sum_{n=0}^\infty \int_0^1 g_n(t)\,dt = \int_0^1 \left(\sum_{n=0}^\infty g_n(t) \right) dt,$$

which is a case of the Lebesgue monotone convergence theorem. □

By Lemma 1, we have

$$\begin{aligned}
S_2 &= \sum_{\ell=1}^\infty \frac{1}{2\ell-1} \int_0^1 \left(\sum_{m=\ell}^\infty (-1)^{m-1} t^{m-1} \right) dt \\
&= \sum_{\ell=1}^\infty \frac{1}{2\ell-1} \int_0^1 \frac{(-t)^{\ell-1}}{1+t}\,dt.
\end{aligned}$$

Since the outer sum is absolutely convergent, we may freely interchange it with the integral:

$$\begin{aligned}
S_2 &= \int_0^1 \left(\sum_{\ell=1}^\infty \frac{1}{2\ell-1} \frac{(-t)^{\ell-1}}{1+t} \right) dt \\
&= \int_0^1 \frac{1}{\sqrt{t}(1+t)} \left(\sum_{\ell=1}^\infty \frac{(-1)^{\ell-1} t^{\ell-1/2}}{2\ell-1} \right) dt \\
&= \int_0^1 \frac{1}{\sqrt{t}(1+t)} \arctan(\sqrt{t})\,dt \\
&= \int_0^1 \frac{2}{1+u^2} \arctan(u)\,du \qquad (u = \sqrt{t}) \\
&= \arctan(1)^2 - \arctan(0)^2 = \frac{\pi^2}{16}.
\end{aligned}$$