A1 Prove that every nonzero coefficient of the Taylor series of

$$(1-x+x^2)e^x$$

about x = 0 is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

A2 Let A be the $n \times n$ matrix whose entry in the *i*-th row and *j*-th column is

$$\frac{1}{\min(i,j)}$$

for $1 \le i, j \le n$. Compute det(*A*).

A3 Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \ge 1$. Compute

$$\prod_{k=0}^{\infty} \left(1 - \frac{1}{a_k} \right)$$

in closed form.

A4 Suppose X is a random variable that takes on only nonnegative integer values, with E[X] = 1, $E[X^2] = 2$, and $E[X^3] = 5$. (Here E[y] denotes the expectation of the random variable Y.) Determine the smallest possible rom Note value of the probability of the event X = 0.

A5 Let

 $P_n(x) = 1 + 2x + 3x^2 + \frac{1}{2}$ Prove that the polynum is $P_j(x)$ and I prime 0 at 1 > 1, which integers j and k

- A6 Let *n* be a positive integer. What is the largest kfor which there exist $n \times n$ matrices M_1, \ldots, M_k and N_1, \ldots, N_k with real entries such that for all *i* and *j*, the matrix product $M_i N_i$ has a zero entry somewhere on its diagonal if and only if $i \neq j$?
- B1 A base 10 over-expansion of a positive integer N is an expression of the form

$$N = d_k 10^k + d_{k-1} 10^{k-1} + \dots + d_0 10^0$$

with $d_k \neq 0$ and $d_i \in \{0, 1, 2, \dots, 10\}$ for all *i*. For instance, the integer N = 10 has two base 10 overexpansions: $10 = 10 \cdot 10^0$ and the usual base 10 expansion $10 = 1 \cdot 10^1 + 0 \cdot 10^0$. Which positive integers have a unique base 10 over-expansion?

- B2 Suppose that f is a function on the interval [1,3] such that $-1 \le f(x) \le 1$ for all x and $\int_1^3 f(x) dx = 0$. How large can $\int_{1}^{3} \frac{f(x)}{r} dx$ be?
- B3 Let A be an $m \times n$ matrix with rational entries. Suppose that there are at least m + n distinct prime numbers among the absolute values of the entries of A. Show that the rank of A is at least 2.
- B4 Show that for each positive integer *n*, all the roots of the polynomial

$$\sum_{k=0}^{n} 2^{k(n-k)} x^k$$

are real numbers.

- B5 In the 75th annual Putnam Games, participants compete at mathematical games. Patniss and Keeta play a game in which they take turns choosing an element from the group of invertible $n \times n$ matrices with entries in the field $\mathbb{Z}/p\mathbb{Z}$ of integers modulo p, where n is a fixed positive integer and p is a fixed prime number. The rules of the game are:
 - (1) A player cannot cro se lement that has been chose weither player on any previous turn.

A sayer can only choose an element that commutes with all previously chosen elements.

(c) A flayer who cannot choose an element on his/her tuen loses the game.

Patniss takes the first turn. Which player has a winning strategy? (Your answer may depend on *n* and *p*.)

B6 Let $f: [0,1] \to \mathbb{R}$ be a function for which there exists a constant K > 0 such that $|f(x) - f(y)| \le K |x - y|$ for all $x, y \in [0, 1]$. Suppose also that for each rational number $r \in [0,1]$, there exist integers a and b such that f(r) =a + br. Prove that there exist finitely many intervals I_1, \ldots, I_n such that f is a linear function on each I_i and $[0,1] = \bigcup_{i=1}^{n} I_i.$