Solutions to the 74th William Lowell Putnam Mathematical Competition
Saturday, December 7, 2013

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A1 Suppose otherwise. Then each vertex v is a vertex for five faces, all of which have different labels, and so the sum of the labels of the five faces incident to v is at least $0+1+2+3+4=10$. Adding this sum over all vertices v gives $3\times 39=117$, since each face's label is counted three times. Since there are 12 vertices, we conclude that $10\times 12\leq 117$, contradiction.

Remark: One can also obtain the desired result by showing that any collection of five faces must contain two faces that share a vertex; it then follows that each label can appear at most 4 times, and so the sum of all labels is at least $4(0+1+2+3+4)=40>39$, contradiction.

A2 Suppose to the contrary that $f(n)=f(m)$ with $n<m$, and let $n\cdot a_1\cdots a_r$, $m\cdot b_1\cdots b_s$ be perfect squares where $n<a_1<\cdots<a_r$, $m<b_1<\cdots<b_s$, a_r,b_s are minimal and $a_r=b_s$. Then $(n\cdot a_1\cdots a_r)\cdot(m\cdot b_1\cdots b_s)$ is also a perfect square. Now eliminate any factor in this product that appears twice (i.e., if $a_i=b_j$ for some i,j, then delete a_i and b_j from this product). The product of what remains must also be a perfect square, but this is now a product of distinct integers, the smallest of which is n and the largest of which is strictly smaller than $a_r=b_s$. This contradicts the minimality of a_r.

Remark: Sequences whose product is a perfect square occur naturally in the *quadratic sieve* algorithm for factoring large integers. However, the behavior of the function $f(n)$ seems to be somewhat erratic. Karl Mahlburg points out the upper bound $f(n)\leq 2n$ for $n\geq 5$, which holds because the interval $(n,2n)$ contains an integer of the form $2m^2$. A trivial lower bound is $f(n)\geq n+p$ where p is the least prime factor of n. For $n=p$ prime, the bounds agree and we have $f(p)=2p$. For more discussion, see `https://oeis.org/A006255`.

A3 Suppose on the contrary that $a_0+a_1y+\cdots+a_ny^n$ is nonzero for $0<y<1$. By the intermediate value theorem, this is only possible if $a_0+a_1y+\cdots+a_ny^n$ has the same sign for $0<y<1$; without loss of generality, we may assume that $a_0+a_1y+\cdots+a_ny^n>0$ for $0<y<1$. For the given value of x, we then have

$$a_0x^m+a_1x^{2m}+\cdots+a_nx^{(n+1)m}\geq 0$$

for $m=0,1,\ldots$, with strict inequality for $m>0$. Taking the sum over all m is absolutely convergent and hence valid; this yields

$$\frac{a_0}{1-x}+\frac{a_1}{1-x^2}+\cdots+\frac{a_n}{1-x^{n+1}}>0,$$

a contradiction.

A4 Let $w'_1,\ldots,w'_k$ be arcs such that: w'_j has the same length as w_j; w'_1 is the same as w_1; and w'_{j+1} is adjacent to w'_j (i.e., the last digit of w'_j comes right before the first digit of w'_{j+1}). Since w_j has length $Z(w_j)+N(w_j)$, the sum of the lengths of $w_1,\ldots,w_k$ is $k(Z+N)$, and so the concatenation of $w'_1,\ldots,w'_k$ is a string of $k(Z+N)$ consecutive digits around the circle. (This string may wrap around the circle, in which case some of these digits may appear more than once in the string.) Break this string into k arcs $w''_1,\ldots,w''_k$ each of length $Z+N$, each adjacent to the previous one. (Note that if the number of digits around the circle is m, then $Z+N\leq m$ since $Z(w_j)+N(w_j)\leq m$ for all j, and thus each of $w''_1,\ldots,w''_k$ is indeed an arc.)

We claim that for some $j=1,\ldots,k$, $Z(w''_j)=Z$ and $N(w''_j)=N$ (where the second equation follows from the first since $Z(w''_j)+N(w''_j)=Z+N$). Otherwise, since all of the $Z(w''_j)$ differ by at most 1, either $Z(w''_j)\leq Z-1$ for all j or $Z(w''_j)\geq Z+1$ for all j. In either case, $|kZ-\sum_j Z(w'_j)|=|kZ-\sum_j Z(w''_j)|\geq k$. But since $w_1=w'_1$, we have $|kZ-\sum_j Z(w'_j)|=|\sum_{j=1}^k(Z(w_j)-Z(w'_j))|=|\sum_{j=2}^k(Z(w_j)-Z(w'_j))|\leq\sum_{j=2}^k|Z(w_j)-Z(w'_j)|\leq k-1$, contradiction.

A5 Let $A_1,\ldots,A_m$ be points in $\mathbb{R}^3$, and let $\hat{n}_{ijk}$ denote a unit vector normal to $\Delta A_iA_jA_k$ (unless A_i,A_j,A_k are collinear, there are two possible choices for $\hat{n}_{ijk}$). If $\hat{n}$ is a unit vector in $\mathbb{R}^3$, and $\Pi_{\hat{n}}$ is a plane perpendicular to $\hat{n}$, then the area of the orthogonal projection of $\Delta A_iA_jA_k$ onto $\Pi_{\hat{n}}$ is $\mathrm{Area}(\Delta A_iA_jA_k)|\hat{n}_{ijk}\cdot\hat{n}|$. Thus if $\{a_{ijk}\}$ is area definite for $\mathbb{R}^2$, then for any $\hat{n}$,

$$\sum a_{ijk}\mathrm{Area}(\Delta A_iA_jA_k)|\hat{n}_{ijk}\cdot\hat{n}|\geq 0.$$

Note that integrating $|\hat{n}_{ijk}\cdot\hat{n}|$ over $\hat{n}\in S^2$, the unit sphere in $\mathbb{R}^3$, with respect to the natural measure on S^2 gives a positive number c, which is independent of $\hat{n}_{ijk}$ since the measure on S^2 is rotation-independent. Thus integrating the above inequality over $\hat{n}$ gives $c\sum a_{ijk}\mathrm{Area}(\Delta A_iA_jA_k)\geq 0$. It follows that $\{a_{ijk}\}$ is area definite for $\mathbb{R}^3$, as desired.

Remark: It is not hard to check (e.g., by integration in spherical coordinates) that the constant c occurring above is equal to 2π. It follows that for any convex body C in $\mathbb{R}^3$, the average over $\hat{n}$ of the area of the projection of C onto $\Pi_{\hat{n}}$ equals $1/4$ of the surface area of C.

More generally, let C be a convex body in $\mathbb{R}^n$. For $\hat{n}$ a unit vector, let $\Pi_{\hat{n}}$ denote the hyperplane through the origin perpendicular to $\hat{n}$. Then the average over $\hat{n}$ of the