volumes sum to V. Then on one hand,

$$\sum_{i=1}^{n} \frac{4}{3}\pi r_i^3 = V.$$

On the other hand, the intersection of a ball of radius r with the plane containing F is a disc of radius at most r, which covers a piece of F of area at most πr^2; therefore

$$\sum_{i=1}^{n} \pi r_i^2 \geq A.$$

By writing n as $\sum_{i=1}^{n} 1$ and applying Hölder's inequality, we obtain

$$nV^2 \geq \left(\sum_{i=1}^{n} \left(\frac{4}{3}\pi r_i^3 \right)^{2/3} \right)^3 \geq \frac{16}{9\pi} A^3.$$

Consequently, any value of $c(P)$ less than $\frac{16}{9\pi}A^3$ works.

B3 The answer is yes. We first note that for any collection of m days with $1 \leq m \leq 2n-1$, there are at least m distinct teams that won a game on at least one of those days. If not, then any of the teams that lost games on all of those days must in particular have lost to m other teams, a contradiction.

If we now construct a bipartite graph whose vertices are the $2n$ teams and the $2n-1$ days, with an edge linking a day to a team if that team won their game on that day, then any collection of m days is connected to a total of at least m teams. It follows from Hall's Marriage Theorem that one can match the $2n-1$ days with $2n-1$ distinct teams that won on their respective days, as desired.

B4 **First solution.** We will show that the answer is yes. First note that for all $x > -1$, $e^x \geq 1+x$ and thus

$$x \geq \log(1+x). \tag{2}$$

We next claim that $a_n > \log(n+1)$ (and in particular that $a_n - \log n > 0$) for all n, by induction on n. For $n = 0$ this follows from $a_0 = 1$. Now suppose that $a_n > \log(n+1)$, and define $f(x) = x + e^{-x}$, which is an increasing function in $x > 0$; then

$$\begin{aligned} a_{n+1} &= f(a_n) > f(\log(n+1)) \\ &= \log(n+1) + 1/(n+1) \geq \log(n+2), \end{aligned}$$

where the last inequality is (2) with $x = 1/(n+1)$. This completes the induction step.

It follows that $a_n - \log n$ is a decreasing function in n: we have

$$\begin{aligned} &(a_{n+1} - \log(n+1)) - (a_n - \log n) \\ &= e^{-a_n} + \log(n/(n+1)) \\ &< 1/(n+1) + \log(n/(n+1)) \leq 0, \end{aligned}$$

where the final inequality is (2) with $x = -1/(n+1)$. Thus $\{a_n - \log n\}_{n=0}^{\infty}$ is a decreasing sequence of positive numbers, and so it has a limit as $n \to \infty$.

Second solution. Put $b_n = e^{a_n}$, so that $b_{n+1} = b_n e^{1/b_n}$. In terms of the b_n, the problem is to prove that b_n/n has a limit as $n \to \infty$; we will show that the limit is in fact equal to 1.

Expanding e^{1/b_n} as a Taylor series in $1/b_n$, we have

$$b_{n+1} = b_n + 1 + R_n$$

where $0 \leq R_n \leq c/b_n$ for some absolute constant $c > 0$. By writing

$$b_n = n + e + \sum_{i=0}^{n-1} R_i,$$

we see first that $b_n \geq n + e$. We then see that

$$\begin{aligned} 0 &\leq \frac{b_n}{n} - 1 \\ &\leq \frac{e}{n} + \sum_{i=0}^{n-1} \frac{R_i}{n} \\ &\leq \frac{e}{n} + \sum_{i=0}^{n-1} \frac{c}{nb_i} \\ &\leq \frac{e}{n} + \sum_{i=0}^{n-1} \frac{c}{n(i+e)} \\ &\leq \frac{e}{n} + \frac{c\log n}{n}. \end{aligned}$$

It follows that $b_n/n \to 1$ as $n \to \infty$.

Remark. This problem is an example of the general principle that one can often predict the asymptotic behavior of a recursive sequence by studying solutions of a sufficiently similar-looking differential equation. In this case, we start with the equation $a_{n+1} - a_n = e^{-a_n}$, then replace a_n with a function $y(x)$ and replace the difference $a_{n+1} - a_n$ with the derivative $y'(x)$ to obtain the differential equation $y' = e^{-y}$, which indeed has the solution $y = \log x$.

B5 Define the function

$$f(x) = \sup_{s \in \mathbb{R}} \{x \log g_1(s) + \log g_2(s)\}.$$

As a function of x, f is the supremum of a collection of affine functions, so it is convex. The function $e^{f(x)}$ is then also convex, as may be checked directly from the definition: for $x_1, x_2 \in \mathbb{R}$ and $t \in [0,1]$, by the weighted AM-GM inequality

$$\begin{aligned} te^{f(x_1)} + (1-t)e^{f(x_2)} &\geq e^{tf(x_1)+(1-t)f(x_2)} \\ &\geq e^{f(tx_1+(1-t)x_2)}. \end{aligned}$$

For each $t \in \mathbb{R}$, draw a supporting line to the graph of $e^{f(x)}$ at $x = t$; it has the form $y = xh_1(t) + h_2(t)$ for some $h_1(t), h_2(t) \in \mathbb{R}$. For all x, we then have

$$\sup_{s \in \mathbb{R}} \{g_1(s)^x g_2(s)\} \geq xh_1(t) + h_2(t)$$