B–5 **First solution.** The answer is no. Suppose otherwise. For the condition to make sense, f must be differentiable. Since f is strictly increasing, we must have $f'(x) \geq 0$ for all x. Also, the function $f'(x)$ is strictly increasing: if $y > x$ then $f'(y) = f(f(y)) > f(f(x)) = f'(x)$. In particular, $f'(y) > 0$ for all $y \in \mathbb{R}$.

For any $x_0 \geq -1$, if $f(x_0) = b$ and $f'(x_0) = a > 0$, then $f'(x) > a$ for $x > x_0$ and thus $f(x) \geq a(x - x_0) + b$ for $x \geq x_0$. Then either $b < x_0$ or $a = f'(x_0) = f(f(x_0)) = f(b) \geq a(b - x_0) + b$. In the latter case, $b \leq a(x_0 + 1)/(a+1) \leq x_0 + 1$. We conclude in either case that $f(x_0) \leq x_0 + 1$ for all $x_0 \geq -1$.

It must then be the case that $f(f(x)) = f'(x) \leq 1$ for all x, since otherwise $f(x) > x + 1$ for large x. Now by the above reasoning, if $f(0) = b_0$ and $f'(0) = a_0 > 0$, then $f(x) > a_0 x + b_0$ for $x > 0$. Thus for $x > \max\{0, -b_0/a_0\}$, we have $f(x) > 0$ and $f(f(x)) > a_0 x + b_0$. But then $f(f(x)) > 1$ for sufficiently large x, a contradiction.

Second solution. (Communicated by Catalin Zara.) Suppose such a function exists. Since f is strictly increasing and differentiable, so is $f \circ f = f'$. In particular, f is twice differentiable; also, $f''(x) = f'(f(x))f'(x)$ is the product of two strictly increasing nonnegative functions, so it is also strictly increasing and nonnegative. In particular, we can choose $\alpha > 0$ and $M \in \mathbb{R}$ such that $f''(x) > 4\alpha$ for all $x \geq M$. Then for all $x \geq M$,

$$f(x) \geq f(M) + f'(M)(x - M) + 2\alpha(x - M)^2.$$

In particular, for some $M' > M$, we have $f(x) \geq \alpha x^2$ for all $x \geq M'$.

Pick $T > 0$ so that $\alpha T^2 > M'$. Then for $x \geq T$, $f(x) > M'$ and so $f'(x) = f(f(x)) \geq \alpha f(x)^2$. Now

$$\frac{1}{f(T)} - \frac{1}{f(2T)} = \int_T^{2T} \frac{f'(t)}{f(t)^2}\,dt \geq \int_T^{2T} \alpha\,dt;$$

however, as $T \to \infty$, the left side of this inequality tends to 0 while the right side tends to $+\infty$, a contradiction.

Third solution. (Communicated by Noam Elkies.) Since f is strictly increasing, for some y_0, we can define the inverse function $g(y)$ of f for $y \geq y_0$. Then $x = g(f(x))$, and we may differentiate to find that $1 = g'(f(x))f'(x) = g'(f(x))f(f(x))$. It follows that $g'(y) = 1/f(y)$ for $y \geq y_0$; since g takes arbitrarily large values, the integral $\int_{y_0}^{\infty} dy/f(y)$ must diverge. One then gets a contradiction from any reasonable lower bound on $f(y)$ for y large, e.g., the bound $f(x) \geq \alpha x^2$ from the second solution. (One can also start with a linear lower bound $f(x) \geq \beta x$, then use the integral expression for g to deduce that $g(x) \leq \gamma \log x$, which in turn forces $f(x)$ to grow exponentially.)

B–6 For any polynomial $p(x)$, let $[p(x)]A$ denote the $n \times n$ matrix obtained by replacing each entry A_{ij} of A by $p(A_{ij})$; thus $A^{[k]} = [x^k]A$. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ denote the characteristic polynomial of A. By the Cayley-Hamilton theorem,

$$\begin{aligned}
0 &= A \cdot P(A) \\
&= A^{n+1} + a_{n-1}A^n + \cdots + a_0 A \\
&= A^{[n+1]} + a_{n-1}A^{[n]} + \cdots + a_0 A^{[1]} \\
&= [xp(x)]A.
\end{aligned}$$

Thus each entry of A is a root of the polynomial $xp(x)$.

Now suppose $m \geq n+1$. Then

$$\begin{aligned}
0 &= [x^{m+1-n}P(x)]A \\
&= A^{[m+1]} + a_{n-1}A^{[m]} + \cdots + a_0 A^{[m+1-n]}
\end{aligned}$$

since each entry of A is a root of $x^{m+1-n}P(x)$. On the other hand,

$$\begin{aligned}
0 &= A^{m+1-n} \cdot P(A) \\
&= A^{m+1} + a_{n-1}A^m + \cdots + a_0 A^{m+1-n}.
\end{aligned}$$

Therefore if $A^k = A^{[k]}$ for $m+1-n \leq k \leq m$, then $A^{m+1} = A^{[m+1]}$. The desired result follows by induction on m.

Remark. David Feldman points out that the result is best possible in the following sense: there exist examples of $n \times n$ matrices A for which $A^k = A^{[k]}$ for $k = 1, \ldots, n$ but $A^{n+1} \neq A^{[n+1]}$.