product of the orders of all of its elements cannot be a power of 2.

We may thus consider only abelian 2-groups hereafter. For such a group G, the product of the orders of all of its elements has the form $2^{k(G)}$ for some nonnegative integer G, and we must show that it is impossible to achieve $k(G) = 2009$. Again by the structure theorem, we may write

$$G \cong \prod_{i=1}^{\infty} (\mathbb{Z}/2^i\mathbb{Z})^{e_i}$$

for some nonnegative integers $e_1, e_2, \ldots$, all but finitely many of which are 0.

For any nonnegative integer m, the elements of G of order at most 2^m form a subgroup isomorphic to

$$\prod_{i=1}^{\infty} (\mathbb{Z}/2^{\min\{i,m\}}\mathbb{Z})^{e_i},$$

which has 2^{s_m} elements for $s_m = \sum_{i=1}^{\infty} \min\{i,m\}e_i$. Hence

$$k(G) = \sum_{i=1}^{\infty} i(2^{s_i} - 2^{s_{i-1}}).$$

Since $s_1 \le s_2 \le \cdots$, $k(G)+1$ is always divisible by 2^{s_1}. In particular, $k(G) = 2009$ forces $s_1 \le 1$.

However, the only cases where $s_1 \le 1$ are where all of the e_i are 0, in which case $k(G) = 0$, or where $e_i = 1$ for some i and $e_j = 0$ for $j \neq i$, in which case $k(G) = (i-1)2^i + 1$. The right side is a strictly increasing function of i which equals 1793 for $i = 8$ and 4097 for $i = 9$, so it can never equal 2009. This proves the claim.

Remark. One can also arrive at the key congruence by dividing G into equivalence classes, by declaring two elements to be equivalent if they generate the same cyclic subgroup of G. For $h > 0$, an element of order 2^h belongs to an equivalence class of size 2^{h-1}, so the products of the orders of the elements of this equivalence class is 2^j for $j = h2^{h-1}$. This quantity is divisible by 4 as long as $h > 1$; thus to have $k(G) \equiv 1 \pmod 4$, the number of elements of G of order 2 must be congruent to 1 modulo 4. However, there are exactly $2^e - 1$ such elements, for e the number of cyclic factors of G. Hence $e = 1$, and one concludes as in the given solution.

A–6 We disprove the assertion using the example

$$f(x,y) = 3(1+y)(2x-1)^2 - y.$$

We have $b-a = d-c = 0$ because the identity $f(x,y) = f(1-x,y)$ forces $a = b$, and because

$$c = \int_0^1 3(2x-1)^2\,dx = 1,$$
$$d = \int_0^1 (6(2x-1)^2 - 1)\,dx = 1.$$

Moreover, the partial derivatives

$$\frac{\partial f}{\partial x}(x_0,y_0) = 3(1+y_0)(8x_0-4)$$
$$\frac{\partial f}{\partial y}(x_0,y_0) = 3(2x_0-1)^2 - 1.$$

have no common zero in $(0,1)^2$. Namely, for the first partial to vanish, we must have $x_0 = 1/2$ since $1+y_0$ is nowhere zero, but for $x_0 = 1/2$ the second partial cannot vanish.

Remark. This problem amounts to refuting a potential generalization of the Mean Value Theorem to bivariate functions. Many counterexamples are possible. Kent Merryfield suggests $y\sin(2\pi x)$, for which all four of the boundary integrals vanish; here the partial derivatives are $2\pi y\cos(2\pi x)$ and $\sin(2\pi x)$. Catalin Zara suggests $x^{1/3}y^{2/3}$. Qingchun Ren suggests $xy(1-y)$.

B–1 Every positive rational number can be uniquely written in lowest terms as a/b for a,b positive integers. We prove the statement in the problem by induction on the largest prime dividing either a or b (where this is considered to be 1 if $a = b = 1$). For the base case, we can write $1/1 = 2!/2!$. For a general a/b, let p be the largest prime dividing either a or b; then $a/b = p^k a'/b'$ for some $k \ge 0$ and positive integers a', b' whose largest prime factors are strictly less than p. We now have $a/b = (p!)^k \frac{a'}{(p-1)!^k b'}$, and all prime factors of a' and $(p-1)!^k b'$ are strictly less than p. By the induction assumption, $\frac{a'}{(p-1)!^k b'}$ can be written as a quotient of products of prime factorials, and so $a/b = (p!)^k \frac{a'}{(p-1)!^k b'}$ can as well. This completes the induction.

Remark. Noam Elkies points out that the representations are unique up to rearranging and canceling common factors.

B–2 The desired real numbers c are precisely those for which $1/3 < c \le 1$. For any positive integer m and any sequence $0 = x_0 < x_1 < \cdots < x_m = 1$, the cost of jumping along this sequence is $\sum_{i=1}^{m} (x_i - x_{i-1})x_i^2$. Since

$$\begin{aligned} 1 = \sum_{i=1}^{m} (x_i - x_{i-1}) &\ge \sum_{i=1}^{m} (x_i - x_{i-1})x_i^2 \\ &> \sum_{i=1}^{m} \int_{x_i}^{x_{i-1}} t^2\,dt \\ &= \int_0^1 t^2\,dt = \frac{1}{3}, \end{aligned}$$

we can only achieve costs c for which $1/3 < c \le 1$.

It remains to check that any such c can be achieved. Suppose $0 = x_0 < \cdots < x_m = 1$ is a sequence with $m \ge 1$. For $i = 1, \ldots, m$, let c_i be the cost of the sequence $0, x_i, x_{i+1}, \ldots, x_m$. For $i > 1$ and $0 < y \le x_{i-1}$, the cost of the sequence $0, y, x_i, \ldots, x_m$ is

$$c_i + y^3 + (x_i - y)x_i^2 - x_i^3 = c_i - y(x_i^2 - y^2),$$