Solutions to the 67th William Lowell Putnam Mathematical Competition
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A–1 We change to cylindrical coordinates, i.e., we put $r = \sqrt{x^2+y^2}$. Then the given inequality is equivalent to

$$r^2 + z^2 + 8 \leq 6r,$$

or

$$(r-3)^2 + z^2 \leq 1.$$

This defines a solid of revolution (a solid torus); the area being rotated is the disc $(x-3)^2 + z^2 \leq 1$ in the xz-plane. By Pappus's theorem, the volume of this equals the area of this disc, which is π, times the distance through which the center of mass is being rotated, which is $(2\pi)3$. That is, the total volume is $6\pi^2$.

A–2 Suppose on the contrary that the set B of values of n for which Bob has a winning strategy is finite; for convenience, we include $n = 0$ in B, and write $B = \{b_1, \dots, b_m\}$. Then for every nonnegative integer n not in B, Alice must have some move on a heap of n stones leading to a position in which the second player wins. That is, every nonnegative integer not in B can be written as $b + p - 1$ for some $b \in B$ and some prime p. However, there are numerous ways to show that this cannot happen.

First solution: Let t be any integer bigger than all of the $b \in B$. Then it is easy to write down t consecutive composite integers, e.g., $(t+1)!+2, \dots, (t+1)!+t+1$. Take $n = (t+1)!+t$; then for each $b \in B$, $n - b + 1$ is one of the composite integers we just wrote down.

Second solution: Let $p_1, \dots, p_{2m}$ be any prime numbers; then by the Chinese remainder theorem, there exists a positive integer x such that

$$\begin{aligned} x - b_1 &\equiv -1 \pmod{p_1 p_{m+1}} \\ &\dots \\ x - b_n &\equiv -1 \pmod{p_m p_{2m}}. \end{aligned}$$

For each $b \in B$, the unique integer p such that $x = b + p - 1$ is divisible by at least two primes, and so cannot itself be prime.

Third solution: (by Catalin Zara) Put $b_1 = 0$, and take $n = (b_2 - 1) \cdots (b_m - 1)$; then n is composite because $3, 8 \in B$, and for any nonzero $b \in B$, $n - b_i + 1$ is divisible by but not equal to $b_i - 1$. (One could also take $n = b_2 \cdots b_m - 1$, so that $n - b_i + 1$ is divisible by b_i.)

A–3 We first observe that given any sequence of integers $x_1, x_2, \dots$ satisfying a recursion

$$x_k = f(x_{k-1}, \dots, x_{k-n}) \qquad (k > n),$$

where n is fixed and f is a fixed polynomial of n variables with integer coefficients, for any positive integer N, the sequence modulo N is eventually periodic. This is simply because there are only finitely many possible sequences of n consecutive values modulo N, and once such a sequence is repeated, every subsequent value is repeated as well.

We next observe that if one can rewrite the same recursion as

$$x_{k-n} = g(x_{k-n+1}, \dots, x_k) \qquad (k > n),$$

where g is also a polynomial with integer coefficients, then the sequence extends uniquely to a doubly infinite sequence $\dots, x_{-1}, x_0, x_1, \dots$ which is fully periodic modulo any N. That is the case in the situation at hand, because we can rewrite the given recursion as

$$x_{k-2005} = x_{k+1} - x_k.$$

It thus suffices to find 2005 consecutive terms divisible by N in the doubly infinite sequence, for any fixed N (so in particular for $N = 2006$). Running the recursion backwards, we easily find

$$\begin{aligned} x_1 = x_0 = \cdots = x_{-2004} &= 1 \\ x_{-2005} = \cdots = x_{-4009} &= 0, \end{aligned}$$

yielding the desired result.

A–4 **First solution:** By the linearity of expectation, the average number of local maxima is equal to the sum of the probability of having a local maximum at k over $k = 1, \dots, n$. For $k = 1$, this probability is 1/2: given the pair $\{\pi(1), \pi(2)\}$, it is equally likely that $\pi(1)$ or $\pi(2)$ is bigger. Similarly, for $k = n$, the probability is 1/2. For $1 < k < n$, the probability is 1/3: given the pair $\{\pi(k-1), \pi(k), \pi(k+1)\}$, it is equally likely that any of the three is the largest. Thus the average number of local maxima is

$$2 \cdot \frac{1}{2} + (n-2) \cdot \frac{1}{3} = \frac{n+1}{3}.$$

Second solution: Another way to apply the linearity of expectation is to compute the probability that $i \in \{1, \dots, n\}$ occurs as a local maximum. The most efficient way to do this is to imagine the permutation as consisting of the symbols $1, \dots, n, *$ written in a circle in some order. The number i occurs as a local maximum if the two symbols it is adjacent to both belong to the set $\{*, 1, \dots, i-1\}$. There are $i(i-1)$ pairs of such symbols and $n(n-1)$ pairs in total, so the probability of