Differential Equations

(Concepts + Pastpaper examples)

Valid for :

- Alevels Further Mathematics
- Undergraduate first year maths
- Undergraduate second year maths

Integrating factor.

$$\int \frac{1}{\sqrt{x^2+1}}\,dx \rightarrow \sinh^{-1}(x) \rightarrow \ln\left(x+\sqrt{x^2+1}\right)$$

$$e^{\ln\left(x+\sqrt{x^2+1}\right)} = x+\sqrt{x^2+1}$$

b.) $x+\sqrt{x^2+1}\left[\frac{dy}{dx} + \frac{y}{\sqrt{x^2+1}}\right] = x+\sqrt{x^2+1}\left(\frac{x^2}{x^2+1} - \frac{x}{\sqrt{x^2+1}}\right)$

$$x+\sqrt{x^2+1}\left[\frac{dy}{dx} + \frac{y}{\sqrt{x^2+1}}\right] = \frac{x^3}{x^2+1} + \frac{x^2}{\sqrt{x^2+1}} - \frac{x^2}{\sqrt{x^2+1}} - x$$

$$x+\sqrt{x^2+1}\left[\frac{dy}{dx} + \frac{y}{\sqrt{x^2+1}}\right] = \frac{x^3}{x^2+1} - x$$

replace with identity

$$\frac{d}{dx}\left(x+\sqrt{x^2+1}\right)(y) = \int \frac{x^3}{x^2+1} - x \, dx$$

$$\left(x+\sqrt{x^2+1}\right)y = \int -\frac{x}{x^2+1}\,dx$$

$$\begin{array}{r} x \\ x^2+1 \,\big)\, x^3 \\ x^3+x \\ -x \end{array}$$

$$\left(x+\sqrt{x^2+1}\right)y = \frac{-1}{2}\int \frac{2x}{x^2+1}\,dx$$

$$y\left(x+\sqrt{x^2+1}\right) = \frac{-1}{2}\ln(x^2+1) + c$$

plug in $y = \ln 2$, $x = 0$

$$\ln(2) = c$$

$$\boxed{y = \frac{\ln(2) - \frac{1}{2}\ln(x^2+1)}{x+\sqrt{x^2+1}}}$$

ii) C.F $\quad n^2 + 9 = 0$

$$n = \pm 3i$$

$$y = (A\cos 3t + B\sin 3t)$$

P.I

$$y = ct^2 + dt + e$$

$$\frac{dy}{dt} = 2ct + d$$

$$\frac{d^2y}{dt^2} = 2c$$

$$2c + 9(ct^2 + dt + e) = 3t^2 + 1$$

$$3 = 9c \qquad 9d = 0 \qquad 2c + 9e = 1$$

$$c = \frac{1}{3} \qquad d = 0 \qquad \frac{2}{3} + 9e = 1$$

$$9e = \frac{1}{3}$$

$$e = \frac{1}{27}$$

General Solution $= y = (A\cos 3t + B\sin 3t) + \frac{1}{3}t^2 + \frac{1}{27}$

$$tx = (A\cos 3t + B\sin 3t) + \frac{1}{3}t^2 + \frac{1}{27}$$

plug in $x = \frac{\pi}{9}$, $t = \frac{\pi}{3}$

$$\frac{\pi^2}{27} = -A + \frac{\pi^2}{27} + \frac{1}{27}$$

$$A = \frac{1}{27}$$

differentiate again.

$$t\frac{dx}{dt} + x = -3A\sin 3t + 3B\cos 3t + \frac{2}{3}t$$

$$\frac{2\pi}{9} + \frac{\pi}{9} = 0 - 3B + \frac{2\pi}{9}$$

June/2020/9231/21/22/Q.7

It is given that $x = t^3 y$ and

$$t^3\frac{d^2y}{dt^2} + \left(4t^3 + 6t^2\right)\frac{dy}{dt} + \left(13t^3 + 12t^2 + 6t\right)y = 61e^{\frac{1}{2}t}.$$

(a) Show that

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13x = 61e^{\frac{1}{2}t}. \qquad [4]$$

(b) Find the general solution for y in terms of t. [7]

Answers: **(a)** $\frac{d^2x}{dt^2}+4\frac{dx}{dt}+13x = t^3\frac{d^2y}{dt^2}+6t^2\frac{dy}{dt}+6ty+4t^3\frac{dy}{dt}+12t^2y+13t^3y = 61e^{\frac{1}{2}t}$

(b) $y = t^{-3}e^{-2t}\left(A\cos 3t + B\sin 3t\right) + 4t^{-3}e^{\frac{1}{2}t}$

a.) $x = t^3 y$

$$\frac{dx}{dt} = t^3\frac{dy}{dt} + y\,3t^2$$

$$\frac{d^2x}{dt^2} = t^3\frac{d^2y}{dt^2} + \frac{dy}{dt}3t^2 + y\,6t + 3t^2\frac{dy}{dt}$$

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13x = t^3\frac{d^2y}{dt^2} + 4t^3\frac{dy}{dt} + 6t^2\frac{dy}{dt} + y\,6t + 12yt^2 + 13t^3y$$

$$= t^3\frac{d^2y}{dt^2} + \left(4t^3+6t^2\right)\frac{dy}{dt} + \left(13t^3+12t^2+6t\right)y =$$

hence shown

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13x = 61e^{\frac{1}{2}t}$$

b.) $n^2 + 4n + 13 = 0$

$n = -2 \pm \sqrt{3}i$

$x = e^{-2t}\left(A\cos\sqrt{3}t + B\cos\sqrt{3}t\right)$

$X = Ce^{\frac{1}{2}t}$