$$\lim_{m\to\infty} \frac{a_m}{a_{m+1}} = \lim_{m\to\infty} \frac{1}{\left(1+\frac{2}{m}\right)} \lim_{m\to\infty} (2m+2)(2m+3)$$

$= \frac{1}{(1+0)} (\infty)(\infty) = \infty$

$R = \infty$

the series is everywhere convergent series.

$IOC = (-\infty, \infty)$

Find ROC, $x + \frac{x^2}{2^2} + \frac{2!\,x^3}{3^3} + \frac{3!\,x^4}{4!} + \cdots$

(6) It's of the form $\sum \frac{(n-1)!\,x^n}{n^n}$

$a_n = \frac{(n-1)!}{n^n}$, $a_{n+1} = \frac{n!}{(n+1)^{n+1}}$

$\frac{a_n}{a_{n+1}} = \frac{(n-1)!}{n^n} \times \frac{(n+1)^{n+1}}{n!}$

$= \frac{(n-1)!}{n^n} \frac{(n+1)^n \cdot (n+1)}{n(n-1)!} = \left(\frac{n+1}{n}\right)^n \left(\frac{n+1}{n}\right)$

$= \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right)$

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \lim_{n\to\infty} \left(1+\frac{1}{n}\right)$

$= e(1+0) = e$

$\Rightarrow R = e$

(7) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} (x^n)$. Find ROC

$a_n = \frac{(2n)!}{(n!)^2}$, $a_{n+1} = \frac{(2n+2)!}{((n+1)!)^2}$

$\frac{a_n}{a_{n+1}} = \frac{(2n)!}{(n!)^2} \times \frac{((n+1)!)^2}{(2n+2)!} = \frac{(2n)!}{(2n+2)!} \frac{(n+1)!}{n!} \frac{(n+1)!}{n!}$

$= \frac{1}{(2n+2)(2n+1)} (n+1)(n+1)$

$$\Rightarrow \int_0^x f(t)\,dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1}\left(x^{k+1}\right)$$

Remark :- This thm shows that a power series can be integrated term by term inside its interval of convergence

Abel's Thm :- If $\sum a_n x^n$ is a power series having radius of convergence R & let $f(x) = \sum a_n x^n$ where $-R<x<R$. If $\sum a_n R^n$ is also convergent then $\sum a_n x^n$ is U.C. in $[0,R]$ & $\therefore$ in $]-R+\epsilon, R]$ for $\epsilon>0$, we

Q. Prove that $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(x^{2n+1}\right)$ for $x \in (-1,1)$. Show that equality holds for $x=1$. Use this to find a formula for π. What happens at $x=-1$

Sol^n :- To show $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ $\quad -1 < x \le 1$ (Binomial Expansion)

consider, $(1+x^2)^{-1} = 1 - x^2 + \frac{(-1)(-1-1)}{2!}(x^2)^2 + \frac{(-1)(-1-1)(-1-2)(x^2)^3}{3!} + \ldots$ for $|x|<1$

$(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \ldots \; ; |x|<1$ ——(1)

$\Rightarrow$ ROC of this series is 1. $\because a_n = 1$ or -1

$\Rightarrow$ series is uniformly cgt. in $[-1+\epsilon, 1-\epsilon]$ for each $\epsilon>0 \Rightarrow$ series can be integrated term wise.

Integrating (1) on both sides,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + C$$

Put $x=0$, then $\tan^{-1} 0 = 0 - \frac{0^3}{3} + \frac{0^5}{5} + \ldots + C$

$\boxed{C=0}$