For illustrative purposes, we here also include a proof using the ϵ, δ-definition. Thus, let $\epsilon > 0$. We need to show that there exists a $\delta > 0$ such that

$$|x-a| < \delta \implies |fg(x) - fg(a)| < \epsilon.$$

Note that by adding 0 and the triangle inequality, we have

$$\begin{aligned}
|fg(x) - fg(a)| &= |fg(x) - fg(a) + f(x)g(a) - f(x)g(a)| \\
&\leq |f(x)(g(x) - g(a))| + |g(a)(f(x) - f(a))| \\
&= |f(x)|\,|g(x) - g(a)| + |g(a)|\,|f(x) - f(a)| \\
&= |f(x) - f(a) + f(a)|\,|g(x) - g(a)| + |g(a)|\,|f(x) - f(a)| \\
&\leq (|f(x) - f(a)| + |f(a)|)\,|g(x) - g(a)| + |g(a)|\,|f(x) - f(a)|\,.
\end{aligned}$$

Let us assume that $g(a) \neq 0$; the converse case is easier (and left as an exercise to the reader). Now, since f is continuous at a, we can find $\delta_1 > 0$ such that for all x such that $|x-a| < \delta_1$, we have that $|f(x) - f(a)| < \frac{\epsilon}{2|g(a)|}$. Consequently, if $|x - a| < \delta_1$, we have

$$|fg(x) - fg(a)| \leq \left(\frac{\epsilon}{2\,|g(a)|} + |f(a)|\right)|g(x) - g(a)| + \frac{\epsilon}{2}.$$

Similarly, since g is continuous at a, we can find $\delta_2 > 0$ such that for all x such that $|x-a| < \delta_2$, we have $|g(x) - g(a)| < \frac{\epsilon}{2\left(\frac{\epsilon}{2|g(a)|} + |f(a)|\right)}$. Hence, we can let $\delta = \min\{\delta_1, \delta_2\}$. Then, for all x such that $|x - a| < \delta$, it follows that

$$|fg(x) - fg(a)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which is what we wanted to show. This completes the proof (of ii)). □

The previous proposition ensures us that adding, multiplying, or dividing continuous functions results in continuous functions. It is also a very important result that composing continuous functions results in continuous functions. We next give a precise statement and a proof for this:

Theorem 2.2. *Let I, a, and f be as above, and suppose that f is continuous at a. Suppose moreover that g is an $\mathbb{R}$-valued function defined on some D_g containing $f(I)$ (the image of f), and also that g is continuous at $f(a)$. Then the composite function, $g \circ f$, is continuous at a.*

Proof. We shall give 3 proofs of this result, for illustrative purposes.

Proof 1. Let $h = g \circ f$. Using limits, and the continuity of g and f, we have

$$\lim_{x\to a} h(x) = \lim_{x\to a} g(f(x)) = g(\lim_{x\to a} f(x)) = g(f(a)) = h(a),$$

which shows that h is continuous at a. Here we have used the continuity of g in the second equality, and the continuity of f in the third.