1.G. Parametrization of $\mathcal{L}(\mathbb{F}^n, V)$ by lists of n vectors *There is a natural bijection $V^n \to \mathcal{L}(\mathbb{F}^n, V)$. It maps a list $u = (u_1, \dots, u_n) \in V^n$ to a linear map*

$$T_u : \mathbb{F}^n \to V : (x_1, \dots, x_n) \mapsto \sum_{i=1}^{n} x_i u_1.$$

The inverse map maps $T : \mathbb{F}^n \to V$ to the list $(T(\mathfrak{e}_1), \dots T(\mathfrak{e}_n))$, where $\mathfrak{e}_1, \dots, \mathfrak{e}_n \in \mathbb{F}^n$ are the standard basis vectors. $\square$

1.H. Self-duality to the coordinate space: $(\mathbb{F}^n)^{\checkmark} = \mathbb{F}^n$.

Indeed, according to 1.G, we have a bijection $(\mathbb{F}^n)^{\checkmark} = \mathcal{L}(\mathbb{F}^n, \mathbb{F}) \to \mathbb{F}^n$. A covector $\varphi : \mathbb{F}^n \to \mathbb{F}$ corresponds to the list $(\varphi(\mathfrak{e}_1), \dots, \varphi(\mathfrak{e}_n)) \in \mathbb{F}^n$, which can be an arbitrary element of $\mathbb{F}^n$. Verify that this bijection is linear. $\square$

The values $\varphi_1 = \varphi(\mathfrak{e}_1), \dots, \varphi_1 = \varphi(\mathfrak{e}_n)$ of a functional φ on the standard basis vectors $\mathfrak{e}_1, \dots, \mathfrak{e}_n$ can be considered as coordinates of φ in $(\mathbb{F}^n)^{\checkmark}$.

The basis $\mathfrak{e}^1, \dots, \mathfrak{e}^n$ of $(\mathbb{F}^n)^{\checkmark}$ corresponding to these coordinates is defined by formulas $\mathfrak{e}^j(x_1, \dots, x_n) = x_j$.

Indeed, for any $\varphi \in (\mathbb{F}^n)^{\checkmark}$ and $x = (x_1, \dots, x_n) \in \mathbb{F}^n$ we have

$$\begin{aligned} \varphi(x) &= \varphi\Big(\sum_{i=1}^{n} x_i \mathfrak{e}_i\Big) \\ &= \sum_{i=1}^{n} x_i \varphi(\mathfrak{e}_i) \\ &= \sum_{i=1}^{n} \mathfrak{e}^i(x_1, \dots, x_n)\varphi(\mathfrak{e}_i) \\ &= \sum_{i=1}^{n} \varphi(\mathfrak{e}_i)\mathfrak{e}^i(x) = \sum_{i=1}^{n} \varphi_i \mathfrak{e}^i. \end{aligned}$$

Thus $\varphi = \sum_{i=1}^{n} \varphi_i \mathfrak{e}^i$. $\square$

In particular,

$$\mathfrak{e}^j(\mathfrak{e}_i) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} .$$

Here it is convenient to use the Kronecker delta symbol, which is defined by formula

$$\delta_i^j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

With the Kronecker delta the relation between $\mathfrak{e}_i$ and e^j looks as follows: $\mathfrak{e}^j(e_i) = \delta_i^j$.

index appears twice once as lower and once as upper index). For example, formula $x_i \mathfrak{e}^i$ should be understood as $\sum_i x_i \mathfrak{e}^i$. The range of summation is determined from the context.

We will use this skipping of a summation sign cautiously, repeating the same formulas with the summation sign in order to reduce the risk of confusion, until the reader will get comfortable with the shorthand notation and appreciate its flexibility and convenience.

Recall that entries of the matrix of a linear map are involved in the following formulas: the image of the basis vector $\mathfrak{e}_j$ under the linear map with matrix (a_{ij}) is $\sum_{i=1}^m a_{ij}\mathfrak{e}_i$ and the ith coordinate of the image of vector $(x_1, \dots, x_n)$ is $\sum_{j=1}^n a_{ij}x_j$. The first formula suggests to raise the first index of the entry a_{ij}. Then it would take the shape $\sum_{i=1}^m a^i_j \mathfrak{e}_i$ or even $a^i_j \mathfrak{e}_i$ (by skipping the summation sign). In the second formula we have to raise, first, the index at x_j, as it was stated above, and then raising the first index at the matrix entry would make it perfect: $\sum_{j=1}^n a^i_j x^j$. Again, we can skip the summation sign and shorthand $\sum_{j=1}^n a^i_j x^j$ till $a^i_j x^j$.

Thus, in matrices that we met so far, the index numerating lines should be raised to the upper position, while the index numerating rows should be left in the lower position. Then skipping sigmas makes notation very similar to matrix notation: we write $a^i_j x^j$ instead of AX, and, say, matrix expression XAY for $\langle x|Ty\rangle$ that we discussed in 1.10 turns to $x_i a^i_j y^j$ where double summation (both over i and j) is understood. However this similarity falls short when the number of indices increases. It could be preserved if one could use high dimensional matrices.

2. Tensors

2.1. Polylinear maps

Let $V_1, \dots V_n$, W be vector spaces over a field $\mathbb{F}$. A map

$$F : V_1 \times \dots \times V_n \to W : (v_1, \dots, v_n) \mapsto F(v_1, \dots, v_n)$$

is said to be *polylinear* or *multilinear*, if it is linear as a function of each of its arguments, when the other arguments are fixed. In other words,

$$F(v_1, \dots, v_{i-1}, x + y, v_{i+1}, \dots, v_n) = \\ F(v_1, \dots, v_{i-1}, x, v_{i+1}, \dots, v_n) + F(v_1, \dots, v_{i-1}, y, v_{i+1}, \dots, v_n),$$

$$F(v_1, \dots, v_{i-1}, av_i, v_{i+1}, \dots, v_n) = aF(v_1, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_n)$$

for $i = 1, \dots, n$, $a \in \mathbb{F}$. If $W = \mathbb{F}$, a polylinear map is called also a *polylinear function*, or *polylinear functional*, or *polylinear form*.

The set of all polylinear maps $V_1 \times \cdots \times V_n \to W$ is denoted by $\mathcal{L}(V_1, \ldots, V_n; W)$. This is a subspace of the vector space of all maps $V_1 \times \cdots \times V_n \to W$.

2.2. Tensor algebra of a vector space

Let V be a finite dimensional vector space over $\mathbb{F}$. A polylinear functional

$$T : \underbrace{V \times \cdots \times V}_{p \text{ times}} \times \underbrace{V^{\checkmark} \times \cdots \times V^{\checkmark}}_{q \text{ times}} \to \mathbb{F}$$

is called a *tensor* on V of *type* (p, q) and *order* or *valency* $p+q$. It is also said to be a mixed tensor p times *covariant* and q times *contravariant*.

Denote by $\operatorname{Tens}_p^q(V)$ the set of all tensors on a vector space V of type (p, q). As a subspace of $\mathcal{L}(V, \ldots, V, V^{\checkmark}, \ldots, V^{\checkmark}; \mathbb{F})$, $\operatorname{Tens}_p^q(V)$ is a vector space over the same ground field $\mathbb{F}$ as V. If one of the numbers p and q is zero, it is not mentioned in the notation $\operatorname{Tens}_p^q(V)$. Then we write $\operatorname{Tens}_p(V)$ or $\operatorname{Tens}^q(V)$.

Special cases:

- A tensor $V \to \mathbb{F}$ of type $(1, 0)$ is a covector. Thus $\operatorname{Tens}_1(V) = V^{\checkmark}$.
- A tensor $V^{\checkmark} \to \mathbb{F}$ of type $(0, 1)$ is an element of the double dual space $(V^{\checkmark})^{\checkmark}$, and, via the canonical identification of $(V^{\checkmark})^{\checkmark}$ with V, this is a vector. Thus $\operatorname{Tens}^1(V) = V$.
- A tensor $V \times V \to \mathbb{F}$ of type $(2, 0)$ is a bilinear form on V.
- A tensor $V \times V^{\checkmark} \to \mathbb{F}$ of type $(1, 1)$ defines (and is defined by) a linear map $V \to (V^{\checkmark})^{\checkmark} = V$, thus it is identified with an operator $V \to V$. Therefore $\operatorname{Tens}_1^1(V) = \mathcal{L}(V)$.

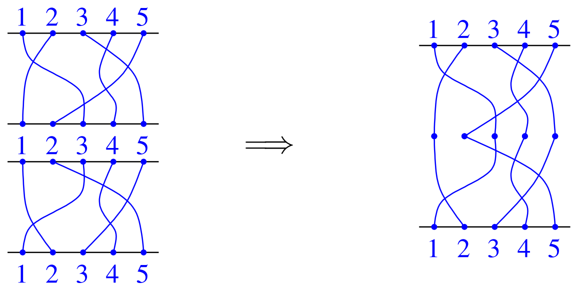
2.3. Coordinates in the spaces of tensors

Let $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ be a basis in a vector space V and $\mathfrak{e}^1, \ldots \mathfrak{e}^n$ be the dual basis in $V^{\checkmark}$. Consider a tensor $T : V^p \times (V^{\checkmark})^q \to \mathbb{F}$. It is defined by its values on lists of base vectors

$$T(\mathfrak{e}_{i_1}, \ldots, \mathfrak{e}_{i_p}, \mathfrak{e}^{j_1}, \ldots, \mathfrak{e}^{j_q}) = T_{i_1, \ldots, i_p}^{j_1, \ldots, j_q}$$

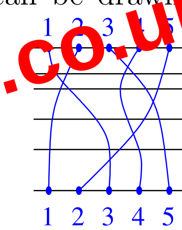
These values are called *coordinates* of T. A tensor of type (p, q) on a vector space of dimension n has n^{p+q} coordinates. A tensor, as a polylinear function on vectors $v_1, \ldots, v_p$ and covectors $u^1, \ldots, u^q$ is determined by

A picture for the composition $\sigma_1 \circ \sigma_1$ of permutations σ_0 and σ_2 can be obtained from the pictures for σ_0 and σ_2 by drawing them one over the other as follows.



3.A. Theorem. *Any permutation can be presented as a composition of transpositions.*

Proof. On a picture of arbitrary permutation, arcs can be drawn in such a way that no two intersection points of the arcs were on the same horizontal line. Then they can be separated from each other by horizontal lines. This gives a desired decomposition of the permutation into a composition of permutations each of which is presented by a picture with one intersection point. Those permutations are transpositions.



□

The arcs, which start at points i and j with $i < j$ and finish at $\sigma(i)$ and $\sigma(j)$, must intersect if $\sigma(i) > \sigma(j)$. They may intersect in several points, but the parity of the number of points depends on the mutual position of $\sigma(i)$ and $\sigma(j)$. Namely, if $\sigma(i) > \sigma(j)$, then the number of intersection points is odd, if $\sigma(i) < \sigma(j)$, then it is even.

A permutation which is a composition of odd number of transpositions is said to be ***odd***, otherwise it is said to be ***even***. The ***sign*** $\operatorname{sign} \sigma$ of a permutation σ is defined to be -1 if σ is odd and $+1$ if σ is even.

3.2. Symmetric tensors

A polylinear form $T : V^k \to \mathbb{F}$ is called ***symmetric*** if its values are not affected by any permutations of the arguments. In other words, T is symmetric, if, for any permutation $\sigma : \{1, \dots, k\} \to \{1, \dots, k\}$ and any $v_1, \dots, v_k \in V$,

$$T(v_1, \dots, v_k) = T(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$